SKOLIAD No. 103

Robert Bilinski

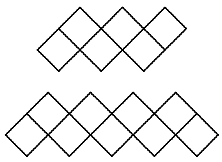
Please send your solutions to the problems in this edition by ***1 February, 2008***. A copy of **MATHEMATICAL MAYHEM Vol. 5** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We received solutions to Skoliad 96 and 97 from Govinda Murali, student, St.Joseph's Public School, Cherthala, India, which arrived too late to be considered for publishing.

In this issue, we present the Team Questions from the 6th annual CNU Regional Mathematics Contest. We thank Ron Persky, C.N.U., Newport News, VA.

6th Annual CNU Regional High School Mathematics Contest (2005)

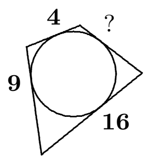
1. To the right are two zigzag shapes made from identical little squares 1 cm on a side. The first shape has 6 squares and a perimeter of 14 cm. The second has 9 squares and a perimeter of 20 cm. What is the perimeter of the zigzag shape with 35 squares?



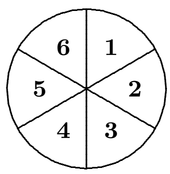
2. Three cards each have one of the digits from 1 through 9 written on them. When the three cards are arranged in some order, they make a three digit number. The largest number that can be made plus the second largest number that can be made is 1233. What is the largest number that can be made?

3. You begin counting on your left hand starting with the thumb, then the index finger, the middle finger, the ring finger, then the little finger, and back to the thumb, and so on. What is the 2005th finger you count?

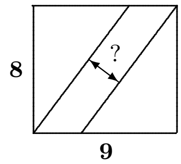
4. A quadrilateral circumscribes a circle. Three of its sides have length 4, 9, and 16 cm, as shown. What is the length in cm of the fourth side?



5. A pizza is cut into six pie-shaped pieces. Trung can choose any piece to eat first, but after that, each piece he chooses must have been next to a piece that has already been eaten (to make it easy to get out of the pan). In how many different orders could he eat the six pieces?



6. The picture shows an 8×9 rectangle cut into three pieces by two parallel slanted lines. The three pieces all have the same area. How far apart are the slanted lines?



7. Find a positive integer N so that there are exactly 25 integers x satisfying $2 \leq \dfrac{N}{x} \leq 5$.

8. Amy, Bart, and Carol ate some carrot sticks. Amy ate half the number that Bart ate, plus one third the number that Carol ate, plus one. Bart ate half the number that Carol ate, plus one-third the number that Amy ate, plus two. Carol ate half the number that Amy ate, plus one-third the number that Bart ate, plus three. How many carrot sticks did they eat altogether?

9. A motorized column is advancing over flat country at the rate of 15 kilometres per hour. The column is 1 kilometre long. A dispatch rider is sent from the rear to the front on a motorcycle travelling at a constant speed. He returns immediately at the same speed and his total time is 3 minutes. How fast is he going?

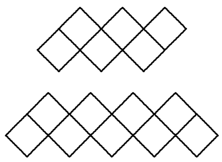
10. Find the remainder when the polynomial $x+x^3+x^9+x^{27}+x^{81}+x^{243}$ is divided by x^2-1.

11. Determine the perimeter of a right triangle with hypotenuse H and area A.

12. When a positive integer n is divided by 3, the remainder is 1. When $n+1$ is divided by 2, the remainder is 1. What is the remainder when $n-1$ is divided by 6?

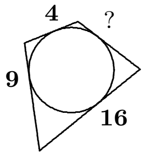
6$^{\text{ième}}$ concours CNU Régional de Mathématiques Secondaires (2005)

1. On retrouve à droit deux formes en zigzag fait de tuiles carrées identiques de 1cm de côté. La première forme a 6 carrés et un périmètre de 14 cm. La seconde a 9 carrés et un périmètre de 20 cm. Quel est le périmètre du zigzag ayant 35 carrés ?

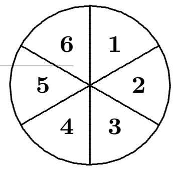


2. Trois cartes ont chacune un des chiffres de 1 à 9 écrit dessus. Quand les trois cartes sont arrangées dans un ordre, elles forment un nombre à trois chiffres. Le plus grand de ces nombres additionné du deuxième plus grand donne 1233. Quel est le plus grand nombre qui peut être fait ?

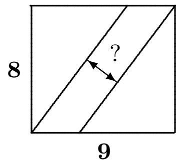
3. On commence à compter sur la main gauche avec le pouce, l'index, le majeur, l'annulaire, l'auriculaire puis on revient au pouce et ainsi de suite. Quel est le $2005^{\text{ième}}$ doigt utilisé ?



4. Un quadrilatère circonscrit un cercle. Trois de ses côtés ont pour longueur 4, 9 et 16 cm comme illustré. Quel est la longueur du quatrième côté ?



5. Une pizza est coupée en six morceaux. Trung peut choisir n'importe quel morceau pour commencer, mais après cela, chaque morceau choisi doit être à côté d'un morceau qui a déjà été choisi au préalable (pour simplifier la sortie du plat). De combien de manières peut-il manger sa pizza ?



6. L'image montre un rectangle 8×9 coupé en trois pièces par deux lignes parallèles obliques. Les trois morceaux ont tous la même aire. Quelle est la distance qui sépare les lignes obliques ?

7. Trouver un entier positif N tel qu'exactement 25 entiers x satisfont à $2 \leq \dfrac{N}{x} \leq 5$.

8. Amy, Bart et Carol ont mangé des mini-carottes. Amy a mangé la moitié du nombre à Bart, plus un tiers du nombre à Carol plus un. Bart a mangé la moitié du nombre à Carol, plus un tiers du nombre à Amy plus deux. Carol a mangé le même nombre qu'Amy plus un tiers du nombre à Bart plus trois. Combien de mini-carottes ont-ils mangé ensemble ?

9. Une colonne motorisée est en train d'avancer en terrain plat au rythme de 15 kilomètres par heure. La colonne est longue de 1 kilomètre. Un courrier est envoyé de l'arrière vers l'avant sur une motocyclette roulant à vitesse constante. Il retourne immédiatement à la même vitesse et son parcours a duré 3 minutes. Quelle était sa vitesse ?

10. Trouver le reste de la division du polynôme $x+x^3+x^9+x^{27}+x^{81}+x^{243}$ par x^2-1.

11. Déterminer le périmètre d'un triangle rectangle ayant une hypothénuse de H et une aire de A.

12. Quand un entier n est divisé par 3, le reste est 1. Quand $n+1$ est divisé par 2, le reste est 1. Quel est le reste quand $n-1$ est divisé par 6 ?

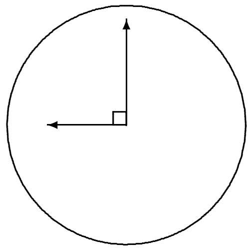
Next we give the solutions to the 2006 Maritime Mathematics Contest [2006 : 481-483].

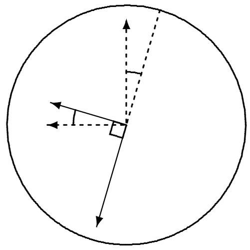
2006 Maritime Mathematics Contest

1. At 9 o'clock, the hour and minute hands on a clock form a right angle. After 9 o'clock, what is the next time at which the clock hands form a right angle?

Official solution, modified by the editor.

The diagrams at right show the hands of a clock at 9:00 (top diagram) and again the next time they form a right angle (bottom diagram). The next time appears to be a little past 9:30.





Let x be the number of minutes after 9:00 the next time the hands form a right angle. The hour hand completes one revolution in 12 hours; hence, in x minutes, it moves through $\frac{x}{12\times 60}$ revolutions. Similarly, since the minute hand takes 60 minutes to complete one revolution, it moves through $x/60$ revolutions in x minutes. In the interval from 9:00 to the required time, the minute hand has moved through exactly half a revolution more than the hour hand. Therefore,

$$\frac{x}{60} = \frac{x}{720} + \frac{1}{2},$$

which yields $11x = 360$, or $x = 32\frac{8}{11}$. Thus, the required time is $32\frac{8}{11}$ minutes after 9:00.

2. For a positive number such as 3.14, we call 3 the *integer part* and 0.14 the *fractional part*. Find a positive number such that the fractional part, the integer part, and the number itself are three consecutive terms

(a) in an arithmetic sequence; (b) in a geometric sequence.

(The sequence a_1, a_2, a_3, a_4, $\ldots$ is called *arithmetic* if there is a number d such that $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots = d$; it is called *geometric* if there is a number $r \neq 0$ such that $a_2/a_1 = a_3/a_2 = a_4/a_3 = \ldots = r$.)

Official solution, modified by the editor.

Let x be a positive number. Let n be its integer part and y its fractional part. Thus, $x = n + y$, where n is an integer and $0 \leq y < 1$.

(a) We want to find x so that y, n, and x are consecutive terms in an arithmetic sequence.

First suppose that $0 < x < 1$. Then $n = 0$ and $x = y$. The numbers y, n, and x are not terms in an arithmetic sequence in this case.

Now assume $x \geq 1$. Then $n \geq 1$ and $y < n \leq x$. In order for y, n, and x to be consecutive terms in an arithmetic sequence, we require

$$n - y \;=\; x - n \;=\; d\,, \tag{1}$$

for some real number d. Since $x = n + y$, we can eliminate x in (1) to get $n - y = y = d$. Then $2y = n$. Since n is a positive integer and $0 \leq y < 1$, this equation is satisfied only when $n = 1$ and $y = \frac{1}{2}$. Thus, the only possible value for x is $x = n + y = 1 + \frac{1}{2} = \frac{3}{2}$.

(b) We want to find x so that y, n, and x are consecutive terms in a geometric sequence. As in part (a), we see that this does not happen if $0 < x < 1$. So we assume $x \geq 1$. Then $n \geq 1$ and $y < n \leq x$.

In order for y, n, and x to be consecutive terms in a geometric sequence, we require

$$\frac{n}{y} \;=\; \frac{x}{n} \;=\; r\,, \tag{2}$$

for some $r \neq 0$. Since $x = n + y$, we can eliminate x in (2) to get

$$\frac{n}{y} \;=\; 1 + \frac{y}{n} \;=\; r\,. \tag{3}$$

Note that y cannot be 0, since this would make n/y undefined. So $0 < y < 1$. Therefore, $n/y > n$ and $1 + (y/n) < 1 + (1/n) \leq 2$ (since $n \geq 1$). Thus,

$$n \;<\; \frac{n}{y} \;=\; 1 + \frac{y}{n} \;\leq\; 2\,.$$

But we know that n is a positive integer. The only possibility satisfying $n < 2$ is $n = 1$.

Setting $n = 1$ in (3), we get $1/y = 1 + y$, which can be rewritten as $y^2 + y - 1 = 0$. By the Quadratic Formula,

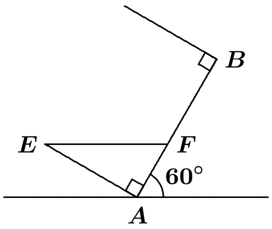
$$y \;=\; \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} \;=\; \frac{-1 \pm \sqrt{5}}{2}\,.$$

Since $y > 0$, we must have $y = \dfrac{-1 + \sqrt{5}}{2}$. Thus, the only possible value for x is $x = n + y = 1 + \dfrac{-1 + \sqrt{5}}{2} = \dfrac{1 + \sqrt{5}}{2}$.

3. A rectangular tank having length 60 cm, width 60 cm, and height 40 cm is filled with water to a depth of 15 cm and rests on a horizontal table. Let A, B, C, and D in cyclic order be the four bottom corners of the tank. Suppose that the edge BC is slowly raised so that the edge AD remains on the table. As water flows out, the tank is raised until the edge AB makes an angle of 60° with the table. The edge BC is then lowered until the tank once again rests on the table. At this point, what is the depth of water in the tank?

Official solution.

Consider the situation when the tank has been raised so that the edge AB makes an angle of 60° with the table. If water did indeed spill from the tank while it was being raised, the water now reaches the level of EF, where E is a top corner of the tank (the corner above A) and F is a point on AB, as shown in the diagram.



(Note that, since the tank has been raised beyond 45°, $|AF|$ is less than $|AE| = 40$, and since $|AB| = 60$, point F does indeed lie on AB.)

Since EF is parallel to the table top, $\angle EFA = 60^\circ$. Hence,

$$|AE| \;=\; |AF|\tan 60^\circ \;=\; |AF|\sqrt{3}\,,$$

or $|AF| = |AE|/\sqrt{3} = 40/\sqrt{3}$.

The volume of water in the tank at this point is equal to the area of $\triangle AEF$ multiplied by 60 cm (the dimension of the tank perpendicular to $\triangle AEF$). Thus, the volume of water is $\frac{1}{2} \times 40 \times \frac{40}{\sqrt{3}} \times 60 = \frac{48000}{\sqrt{3}}$ cm^3. The original volume of water is $15 \times 60 \times 60 = 54000$ cm^3, which is greater than $\frac{48000}{\sqrt{3}}$ cm^3. Therefore, some water does indeed spill from the tank, and the volume of the remaining water is $\frac{48000}{\sqrt{3}}$ cm^3.

Let d be the final depth of water in the tank. Considering the volume of water remaining in the tank, we have $d \times 60 \times 60 = \frac{48000}{\sqrt{3}}$, which implies that $d = \frac{40}{3\sqrt{3}}$. Thus, the final depth of water in the tank is $\frac{40}{3\sqrt{3}}$ cm.

4. Suppose that the positive integers are written in a spiral as shown. Relative to the number 1, where does the number 2006 appear? (For example, 10 appears one unit up and two units to the right of 1.)

7	8	9	10
6	1	2	11
5	4	3	12
	...	14	13

Solution by Natalia Desy, student, Palembang, Indonesia, modified by the editor

The numbers on the diagonal going up to the right are $1, 9, 25, \ldots$, which are the squares of the odd positive integers. The closest odd square to 2006 is $45^2 = 2025$. Since $45 = 2 \times 22 + 1$, we see that 2025 lies 22 units up and 22 units to the right of 1. But 2006 is located $2025 - 2006 = 19$ positions to the left of 2025. Therefore, the position of 2006 is 22 units up and 3 units to the right of 1.

5. A *square pair* is a pair (x, y) of positive integers such that $x+y$ and xy are both perfect squares. For example, $(5, 20)$ is a square pair since $5 + 20 = 25$ and $5 \times 20 = 100$ are both perfect squares. Show that no square pair exists in which one of the numbers is 3.

Official solution, modified by the editor.

Suppose that 3 and x constitute a square pair. Then $3 + x = a^2$ and $3x = b^2$ for some positive integers a and b. Since $3x$ is a perfect square, x

must be of the form $3c^2$ for some positive integer c. Substituting $x = 3c^2$ into $3 + x = a^2$, we obtain $3(1 + c^2) = a^2$; thus, $1 + c^2$ must contain 3 as a factor. We will show that this is not possible.

When we divide c by 3, the remainder must be 0, 1, or 2. If it is 0, then $c = 3k$ for some integer k; this gives $1 + c^2 = 1 + 9k^2$, which does not have 3 as a factor (since 3 is a factor of $9k^2$). If the remainder is 1, then $c = 3k + 1$ for some integer k; this gives $1 + c^2 = 9k^2 + 6k + 2$, which does not have 3 as a factor (since 3 is a factor of $9k^2 + 6k$). If the remainder is 2, then $c = 3k + 2$ for some integer k; this gives $1 + c^2 = (9k^2 + 12k + 3) + 2$, which does not have 3 as a factor (since 3 is a factor of $9k^2 + 12k + 3$).

6. Find all solutions in real numbers x and y for the system of equations:

$$\begin{aligned} 2(x + y - 2) &= y(x - y + 2)\,, \\ x^2(y - 1) + y^2(x - 1) &= xy - 1\,. \end{aligned}$$

Official solution.

Letting $a = x - 1$ and $b = y - 1$, the given equations become

$$\begin{aligned} 2(a + b) &= (b + 1)(a - b + 2)\,, \\ (a + 1)^2 b + (b + 1)^2 a &= (a + 1)(b + 1) - 1\,. \end{aligned}$$

Expanding and simplifying the second equation gives $ab(a + b + 3) = 0$; thus, $a = 0$, $b = 0$, or $a + b = -3$.

If $a = 0$, the first equation becomes $2b = (b + 1)(-b + 2)$, which is equivalent to $b^2 + b - 2 = 0$. Factoring gives $(b + 2)(b - 1) = 0$. Thus, $b = 1$ or $b = -2$.

If $b = 0$, the first equation becomes $2a = a + 2$; thus, $a = 2$.

If $a + b = -3$, we can set $a = -b - 3$ in the first equation to get $2(-3) = (b + 1)(-b - 3 - b + 2)$, which simplifies to $2b^2 + 3b - 5 = 0$. Factoring gives $(2b+5)(b-1) = 0$. Then $b = -\frac{5}{2}$ or $b = 1$. Since $a = -b-3$, we obtain $a = -\frac{1}{2}$ and $a = -4$, respectively.

To summarize, the equations have the following five solutions:

a	b	x	y
0	1	1	2
0	-2	1	-1
2	0	3	1
$-\frac{1}{2}$	$-\frac{5}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$
-4	1	-3	2

An incomplete solution was received.

That brings us to the end of another issue. This month's winner of a past Volume of Mayhem is Natalia Desy. Congratulations, Natalia! Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Please send your solutions to the problems in this edition by **1 January 2008**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M301. *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

The general term of a sequence is $t_n = n^2 + 20$, for $n \geq 1$. Show that for all $n \geq 1$, the greatest common divisor of t_n and t_{n+1} must be a divisor of 81.

M302. *Proposed by Babis Stergiou, Chalkida, Greece.*

A triangle ABC has $\angle ABC = \angle ACB = 40^\circ$. If P is a point in the interior of the triangle such that $\angle PBC = 20^\circ$ and $\angle PCB = 30^\circ$, prove that $BP = BA$.

M303. *Proposed by Neven Jurič, Zagreb, Croatia.*

A curious relation among squares states that the sum of $n+1$ consecutive squares, beginning with the square of $n(2n+1)$, is equal to the sum of the squares of the next n consecutive integers. (For example, when $n = 1$ we have $3^2 + 4^2 = 5^2$, and when $n = 2$ we have $10^2 + 11^2 + 12^2 = 13^2 + 14^2$.) Show that this property holds for any $n \geq 1$.

M304. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be real numbers such that both $a+b+c$ and $ab+bc+ca$ are rational numbers. Show that $a^4+b^4+c^4$ is a rational number if and only if the product abc is a rational number.

M305. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Find all real solutions to the following system of equations:

$$\begin{aligned} \sqrt{x}+\sqrt{y}+\sqrt{z} &= 3\,, \\ x\sqrt{x}+y\sqrt{y}+z\sqrt{z} &= 3\,, \\ x^2\sqrt{x}+y^2\sqrt{y}+z^2\sqrt{z} &= 3\,. \end{aligned}$$

M306. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Find all solutions to the following addition problem, in which each letter represents a distinct digit:

			T	E	N
			T	E	N
		N	I	N	E
	E	I	G	H	T
+	T	H	R	E	E
	F	O	R	T	Y

M301. *Proposé par D.E. Prithwijit, University College Cork, République d'Irlande.*

Soit $t_n = n^2+20$, pour $n \geq 1$, le terme général d'une suite. Montrer que pour tout $n \geq 1$, le plus grand commun diviseur de t_n et t_{n+1} doit être un diviseur de 81.

M302. *Proposé par Babis Stergiou, Chalkida, Grèce.*

Soit ABC un triangle avec $\angle ABC = \angle ACB = 40^\circ$. Si P est un point à l'intérieur du triangle de sorte que $\angle PBC = 20^\circ$ et $\angle PCB = 30^\circ$, montrer que $BP = BA$.

M303. *Proposé par Neven Jurič, Zagreb, Croatie.*

Une curieuse relation entre les carrés de nombres naturels montre que la somme de $n+1$ carrés consécutifs, commençant par le carré de $n(2n+1)$, est égale à la somme des carrés des n entiers consécutifs suivants. (Par exemple, si $n=1$ on a $3^2+4^2=5^2$, et si $n=2$ on a $10^2+11^2+12^2=13^2+14^2$.) Montrer que cette propriété est vraie pour tout $n \geq 1$.

M304. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, b et c trois nombres réels tels que les sommes $a+b+c$ et $ab+bc+ca$ sont des nombres rationnels. Montrer qu'alors $a^4+b^4+c^4$ est un nombre rationnel si et seulement si le product abc est un nombre rationnel.

M305. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Trouver toutes les solutions réelles du système d'équations suivant :

$$\begin{aligned} \sqrt{x}+\sqrt{y}+\sqrt{z} &= 3\,, \\ x\sqrt{x}+y\sqrt{y}+z\sqrt{z} &= 3\,, \\ x^2\sqrt{x}+y^2\sqrt{y}+z^2\sqrt{z} &= 3\,. \end{aligned}$$

M306. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Trouver toutes les solutions du problème d'addition suivant, dans lequel chaque lettre représente un chiffre distinct :

$$\begin{array}{rccccc} & & & T & E & N \\ & & & T & E & N \\ & & N & I & N & E \\ & E & I & G & H & T \\ + & T & H & R & E & E \\ \hline & F & O & R & T & Y \end{array}$$

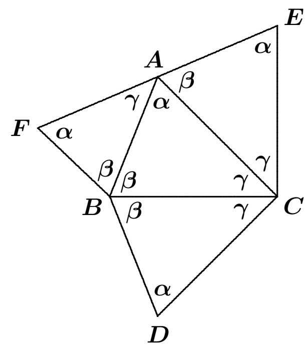
Mayhem Solutions

Some readers pointed out that the solution to M225 which appeared in [2006 : 496–7] *was incorrect. We apologize for this. At this point, we do not have a solution to the problem.*

M251. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let α, β, γ be the angle measures at angles A, B, C, respectively, in $\triangle ABC$. On the sides of $\triangle ABC$, externally, are triangles DBC, EAC, and FBA as in the diagram.

Prove that $AD = EF$ if and only if $\alpha = \pi/2$.



Combination of solutions by Hasan Denker, Istanbul, Turkey; and Jean-David Houle, student, McGill University, Montreal, QC.

It can be seen that triangles ABC and DBC are congruent. From this we conclude that $AB = DB$ and $AC = DC$. Let AG and DG be the altitudes to BC in triangles ABC and DBC, respectively. We can then further conclude that $AG = AB\sin\beta$ and $GD = CD\sin\gamma = AC\sin\gamma$. Consequently,

$$AD \;=\; AG + GD \;=\; AB\sin\beta + AC\sin\gamma\,. \qquad (1)$$

In $\triangle ACE$ we have $AC/\sin\alpha = EA/\sin\gamma$, from which it follows that $EA = AC\sin\gamma/\sin\alpha$. Similarly, in $\triangle AFB$ we have $AF = AB\sin\beta/\sin\alpha$. Since $\alpha+\beta+\gamma = \pi$, we have $EF = EA + AF$, and we obtain

$$EF \;=\; \frac{AC\sin\gamma}{\sin\alpha} + \frac{AB\sin\beta}{\sin\alpha}\,.$$

Using equation (1), we conclude that $EF = AD/\sin\alpha$.

Now, if $\alpha = \pi/2$, then $EF = AD$. Conversely, if $EF = AD$, then $\sin\alpha = 1$, which gives $\alpha = \pi/2$ (since $0 < \alpha < \pi$). Thus, $EF = AD$ if and only if $\alpha = \pi/2$.

Also solved by HOUDA ANOUN, Bordeaux, France; COURTIS G. CHRYSSOSTOMOS and BOTIS A. JIANNHS, Larissa, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VEDULA N. MURTY, Dover, PA, USA; and ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON. There was one incorrect solution submitted.

M252. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let x, y, z be positive real numbers. Prove that

$$\left(\frac{x}{y} + \frac{z}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{y}{z} + \frac{x}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{z}{x} + \frac{y}{\sqrt[3]{xyz}}\right)^2 \;\ge\; 12\,.$$

Essentially the same solution by Mohammed Aassila, Strasbourg, France; Jean-David Houle, student, McGill University, Montreal, QC; D. Kipp Johnson, Beaverton, OR, USA; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA; Alex Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; and Panos E. Tsaoussoglou, Athens, Greece.

Applying the AM–GM Inequality, we obtain the following inequalities:

$$\begin{aligned} \frac{x}{y}+\frac{z}{\sqrt[3]{xyz}} &\geq 2\sqrt{\frac{xz}{y\sqrt[3]{xyz}}}\,, \\ \frac{y}{z}+\frac{x}{\sqrt[3]{xyz}} &\geq 2\sqrt{\frac{yx}{z\sqrt[3]{xyz}}}\,, \\ \frac{z}{x}+\frac{y}{\sqrt[3]{xyz}} &\geq 2\sqrt{\frac{zy}{x\sqrt[3]{xyz}}}\,. \end{aligned}$$

We can then conclude that

$$\begin{aligned} &\left(\frac{x}{y}+\frac{z}{\sqrt[3]{xyz}}\right)^2+\left(\frac{y}{z}+\frac{x}{\sqrt[3]{xyz}}\right)^2+\left(\frac{z}{x}+\frac{y}{\sqrt[3]{xyz}}\right)^2 \\ &\qquad\geq 4\left(\frac{xz}{y\sqrt[3]{xyz}}+\frac{yx}{z\sqrt[3]{xyz}}+\frac{zy}{x\sqrt[3]{xyz}}\right)\,. \end{aligned}$$

Applying the AM–GM Inequality to the right side of this last inequality, we obtain

$$4\left(\frac{xz}{y\sqrt[3]{xyz}}+\frac{yx}{z\sqrt[3]{xyz}}+\frac{zy}{x\sqrt[3]{xyz}}\right) \geq 4\cdot 3\sqrt[3]{\frac{xz}{y\sqrt[3]{xyz}}\cdot\frac{yx}{z\sqrt[3]{xyz}}\cdot\frac{zy}{x\sqrt[3]{xyz}}} = 12\,.$$

Thus,

$$\left(\frac{x}{y}+\frac{z}{\sqrt[3]{xyz}}\right)^2+\left(\frac{y}{z}+\frac{x}{\sqrt[3]{xyz}}\right)^2+\left(\frac{z}{x}+\frac{y}{\sqrt[3]{xyz}}\right)^2 \geq 12\,.$$

Note that equality holds if and only if $x=y=z$.

Also solved by ZAFAR AHMED, BARC, Mumbai, India; ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MIHÁLY BENCZE, Brasov, Romania; QUANG CAO MINH, Nguyen Binh Khiem High School, Vinh Long, Vietnam; SHI CHANGWEI, Xi'an City, Shaan Xi Province, China; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; MATTI LEHTINEN, National Defence College, Helsinki, Finland; VEDULA N. MURTY, Dover, PA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

Bencze actually outlined a solution for a more general problem: If $x_1, x_2, \ldots, x_n$ are positive real numbers, then we have

$$\sum_{\text{cyclic}}\left(\frac{x_1}{x_2}+\frac{x_3}{\sqrt[n]{x_1x_2\cdots x_n}}\right)^\alpha \geq 2^\alpha n$$

for all $\alpha\in(-\infty,0)\cup(1,\infty)$. Of course, the current problem is the case $n=3$ and $\alpha=2$.

M253. *Proposed by Fabio Zucca, Politecnico di Milano, Milano, Italy.*

Consider the set of lattice points $\{(x, y)\}$ where x and y are integers such that $0 \le x \le 7$ and $0 \le y \le 7$. Two points are selected at random from this set. All points have the same probability of being selected and the points need not be distinct. Find the probability that the area of the triangle (possibly degenerate) formed by these two points and the point $(0, 0)$ is an integer (possibly 0).

Solution by Alex Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

Let us denote the area of a triangle XYZ by $[XYZ]$. Recall that for a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$,

$$\begin{aligned}[ABC] &= \frac{1}{2}\left|\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1\end{bmatrix}\right| \\ &= \frac{1}{2}\left|x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3\right| .\end{aligned}$$

For the purpose of our problem, let the two points selected at random be $P(a, b)$ and $Q(c, d)$. If O is the origin, then $[PQO] = \frac{1}{2}|ad - bc|$. In order for $[PQO]$ to be an integer, $|ad - bc|$ must be even, and therefore, ad and bc must have the same parity. This will occur in the following two cases:

Case I. ad and bc are both odd.

This is true if and only if a, b, c, and d are all odd. Since a, b, c, and d belong to the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$, the probability that they are all odd is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$.

Case II. ad and bc are both even.

This is true if and only if a and d are not both odd and b and c are not both odd. The probability that a and d are not both odd is $\frac{3}{4}$, and the same is true for b and c. Therefore, the probability that ad and bc are both even is $\frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$.

Since Case I and Case II are mutually exclusive, the probability that the area of the triangle is an integer is $\frac{1}{16} + \frac{9}{16} = \frac{5}{8}$.

Also solved by HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and D. KIPP JOHNSON, Beaverton, OR, USA. One incorrect solution was also submitted.

M254. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Evaluer la somme $S_{2006} = \sum\limits_{k=1}^{2006} (-1)^k \frac{k^2 - 3}{(k+1)!}$. [On rappelle que $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$; par exemple, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.]

Solution par Jean-David Houle, étudiant, Université McGill, Montréal, QC.

Pour $k \geq 1$, nous savons que

$$\begin{aligned} \frac{k^2-3}{(k+1)!} &= \frac{k(k+1)}{(k+1)!} - \frac{k+1}{(k+1)!} - \frac{2}{(k+1)!} = \frac{1}{(k-1)!} - \frac{1}{k!} - \frac{2}{(k+1)!} \\ &= \left(\frac{1}{(k-1)!} - \frac{1}{(k+1)!}\right) - \left(\frac{1}{k!} + \frac{1}{(k+1)!}\right). \end{aligned}$$

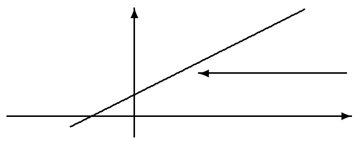
Par manipulations algébriques, on obtient :

$$\begin{aligned} S_{2006} &= \sum_{k=1}^{2006} (-1)^k \frac{k^2-3}{(k+1)!} \\ &= \left(\sum_{k=1}^{2006} \frac{(-1)^k}{(k-1)!} - \sum_{k=1}^{2006} \frac{(-1)^k}{(k+1)!}\right) - \left(\sum_{k=1}^{2006} \frac{(-1)^k}{k!} + \sum_{k=1}^{2006} \frac{(-1)^k}{(k+1)!}\right) \\ &= \left(-1+1+\sum_{k=3}^{2006} \frac{(-1)^k}{(k-1)!} - \sum_{k=1}^{2004} \frac{(-1)^k}{(k+1)!} + \frac{1}{2006!} - \frac{1}{2007!}\right) \\ &\qquad - \left(-1 + \sum_{k=2}^{2006} \frac{(-1)^k}{k!} - \sum_{k=1}^{2005} \frac{(-1)^{k+1}}{(k+1)!} + \frac{1}{2007!}\right) \\ &= \left(\frac{1}{2006!} - \frac{1}{2007!}\right) - \left(-1 + \frac{1}{2007!}\right) \\ &= \frac{1}{2006!} - \frac{2}{2007!} + 1 = \frac{2007-2}{2007!} + 1 = \frac{2005}{2007!} + 1\,. \end{aligned}$$

Autres solutions soumises par RICHARD I. HESS, Rancho Palos Verdes, CA, USA ; et VEDULA N. MURTY, Dover, PA, USA. Deux solutions incorrectes ont aussi été soumises.

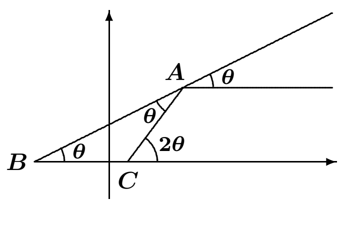
M255. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

The line with slope $\lambda > 0$ acts like a mirror to a ray of light coming along a line parallel to the x–axis. Determine the slope of the reflected ray.



Solution by D. Kipp Johnson, Beaverton, OR, USA.

Let the ray of light hit the mirror at point A, let the x-intercept of the mirror be B, and let the reflected ray of light hit the x-axis at C. If the acute angle formed by the mirror and a horizontal line is θ, then $\lambda = \tan\theta$. Since the angle of incidence and the angle of reflection are equal, we have



$\angle BAC = \angle ABC = \theta$. Then the exterior angle of $\triangle ABC$ at vertex C has measure 2θ, and the slope of the reflected ray is thus

$$\tan 2\theta \;=\; \frac{2\tan\theta}{1-\tan^2\theta} \;=\; \frac{2\lambda}{1-\lambda^2}\,.$$

(If the slope of the mirror is 1, then the reflected ray has undefined slope since it is vertical.)

Also solved by COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; and ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.

M256. *Proposed by the Mayhem Staff.*

Find a quadratic polynomial $f(x)$ such that, if n is a positive integer consisting of the digit 5 repeated k times, then $f(n)$ consists of the digit 5 repeated $2k$ times. (For example, $f(555) = 555555$.)

Solution by Morgan Lynch and Lacey K. Moore, Angelo State University, San Angelo, Texas, USA.

Let $f(x) = ax^2 + bx + c$. Using the given information we obtain the following system of linear equations:

$$\begin{aligned} f(5) &= 25a + 5b + c &=& \; 55\,, \\ f(55) &= 3025a + 55b + c &=& \; 5555\,, \\ f(555) &= 308025a + 555b + c &=& \; 555555\,. \end{aligned}$$

Solving this system, we determine that, $a = \frac{9}{5}$, $b = 2$, and $c = 0$. Thus,

$$f(x) \;=\; \tfrac{9}{5}x^2 + 2x \;=\; x\left(\tfrac{9}{5}x + 2\right).$$

Note that:

$$\begin{aligned} \underbrace{55\cdots 55}_{k \text{ times}} &= 5\left(1 + 10^1 + 10^2 + \cdots + 10^{k-1}\right) \;=\; 5\left(\frac{10^k - 1}{9}\right) \\ &= \tfrac{5}{9}\left(10^k - 1\right). \end{aligned}$$

We can now verify our equation:

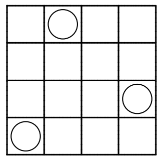
$$\begin{aligned} f\left(\tfrac{5}{9}(10^k - 1)\right) &= \left(\tfrac{5}{9}(10^k - 1)\right)\left[\tfrac{9}{5}\left(\tfrac{5}{9}(10^k - 1)\right) + 2\right] \\ &= \tfrac{5}{9}(10^k - 1)(10^k + 1) \;=\; \tfrac{5}{9}(10^{2k} - 1) \;=\; \underbrace{55\cdots 55}_{2k \text{ times}}\,. \end{aligned}$$

Also solved by HOUDA ANOUN, Bordeaux, France; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; AMY HOLLINGER and CORRIE MEYER, Southeast Missouri State University in Cape Girardeau, Missouri, USA; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; D. KIPP JOHNSON, Beaverton, OR, USA; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina; and ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON. One incorrect solution was also submitted.

Problem of the Month

Ian VanderBurgh

This month's problem involves probability.

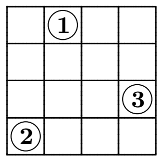


Problem (2007 Euclid Contest)
In the 4×4 grid shown, three coins are randomly placed in different squares. Determine the probability that no two coins lie in the same row or column.

As someone once suggested to me, many probability problems are just combinatorial (counting) problems where you divide by the size of the sample space whenever you want to get a probability. This problem, in particular, boils down to counting the possibilities correctly.

Before actually solving the problem, let's consider how to count the ways of placing the coins on the grid. For the moment, we will require only that no two coins be placed on the same square, without worrying about whether they are in different rows or columns. The number of ways of placing the coins depends on whether the coins are considered to be *distinguishable*. That is, can we tell them apart or are they identical?

First suppose the coins are distinguishable. We will use numbers to refer to them. Coin 1 is the one that is placed first on the grid, followed by coin 2, then coin 3. Coin 1 may be placed anywhere, which means there are 16 possible squares for it. For each of these placements of coin 1, there are 15 open squares remaining in which coin 2 may be placed, giving $16 \cdot 15$ ways of placing the first two coins. For each of these ways, there are 14 squares in which coin 3 may be placed, giving $16 \cdot 15 \cdot 14$ ways of placing all three coins. The figure at right shows one way.

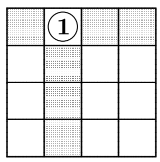


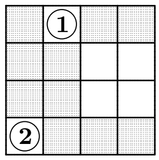
Now suppose the coins are indistinguishable. In this case, they have no numbers. What we want to count now is the number of *configurations* of the coins once they have all been placed on the grid, without regard for the order in which they are placed. Since there are 3 coins and 16 squares, the number of possible configurations is $\binom{16}{3} = \frac{16(15)(14)}{3!} = 560$. This is just our answer for the case where the coins are distinguishable divided by $3!$, the number of ways of rearranging the coins among themselves.

Now let's solve the problem.

Solution 1: Assume the coins are distinguishable. In how many ways can they be placed on the grid so that no two coins are in the same row or column?

There are 16 possible squares for coin 1. Once it has been placed, there are 3 rows and 3 columns remaining that do not contain coin 1, giving $3 \cdot 3 = 9$ squares where coin 2 may be put (the 9 white squares in the figure at right). Thus, there are $16 \cdot 9$ ways of placing the first two coins in different rows and columns.





Once the first two coins have been placed, there are 2 rows and 2 columns remaining that do not contain a coin. Thus, there are $2 \cdot 2 = 4$ squares in which coin 3 may be placed (these are the 4 white squares in the figure at right.) Altogether, there are $16 \cdot 9 \cdot 4$ ways of putting the three coins in different rows and columns.

Since the total number of ways of placing the three coins on the grid is $16 \cdot 15 \cdot 14$ (as we saw earlier), the probability that the three coins are placed in different rows and columns is $\frac{16 \cdot 9 \cdot 4}{16 \cdot 15 \cdot 14} = \frac{6}{35}$.

We can actually compute probabilities at each stage of the calculation instead of waiting until the end. The probability of placing the first two coins in different rows and columns is $\frac{16 \cdot 9}{16 \cdot 15} = \frac{3}{5}$. Given that the first two coins are in different rows and columns, the probability of placing the third coin in a different row and column from each of the first two is $\frac{4}{14} = \frac{2}{7}$ (since there are 4 acceptable squares out of 14 open squares). The probability of placing all three coins in different rows and columns is then $\frac{3}{5} \cdot \frac{2}{7} = \frac{6}{35}$.

Solution 2: This time, we regard the coins as indistinguishable. We will find the number of configurations for the coins in which the coins are in different rows and columns. This could be done by using our counting method for the case where the coins are distinguishable and then dividing by $3!$, but here is a way to count the configurations directly instead.

First, pick the 3 rows in which the coins will be put. There are $\binom{4}{3} = 4$ ways to do this. In the topmost row of these 3 rows, there are 4 possible squares for a coin. In the middle row of these rows, there are 3 possible squares for a coin (since it can't be in the same column as the coin in the topmost row). In the bottom row of these rows, there are 2 possible squares for a coin (since it can't be in the same column as either of the other two coins). Thus, there are $4 \cdot 4 \cdot 3 \cdot 2 = 96$ configurations in which no two coins are in the same row or column.

Finally, the required probability is $\frac{96}{560} = \frac{6}{35}$.

This problem has an interesting history. The initial version asked the same question for 3 coins on a 5×5 grid. (Can you solve this version?) During the development of the 2007 Euclid Contest, the problem was changed to the following problem before being changed back to its original form:

Three *different* numbers are chosen from the set

$$\{11, 12, 13, 14, 21, 22, 23, 24, 31, 32, 33, 34, 41, 42, 43, 44\}\,.$$

What is the probability that no two of these numbers have the same units digit or the same tens digit?

This problem seemed quite a lot harder than the problem with the coins, which is strange, as it is actually the same problem! Can you see why?

You might like to try solving a more general problem where k coins are placed on an $n \times n$ grid (with $k \leq n$, of course).

Note. The author wishes to acknowledge the contributions of Bruce Crofoot, Associate Editor, in the preparation of this column.

Pólya's Paragon

Greatest Common Divisors

Ian VanderBurgh

Most of us learned about greatest common divisors (gcd's) in elementary school when we first learned about prime numbers and prime factorizations. (Remember those prime factorization trees?) We used greatest common divisors again when we learned to add fractions. Since then, however, we have probably forgotten most of what we learned! Here is a refresher on gcd's along with some related calculations and manipulations.

Definition. If a and b are integers that are not both 0, the *greatest common divisor* of a and b, denoted $\gcd(a,b)$, is the largest positive integer that divides exactly into both a and b.

In other words, $\gcd(a,b)$ is the greatest of all the common divisors of a and b. (Don't you wish that all mathematical definitions made this much sense?) To emphasize, d is a *divisor* of a if d divides exactly into a (that is, if $a = qd$ for some integer q). To tidy up a loose end, we say that $\gcd(0,0) = 0$. (Notice here that there is not, in fact, a largest positive integer that divides into both 0 and 0, since every positive integer divides into 0. This means that we either need to ignore this case entirely, or we need to say something special here, as we have done.)

Calculations. Finding the gcd of a pair of integers is not terribly difficult when the integers are small: $\gcd(2,-4) = 2$, $\gcd(3,5) = 1$, and $\gcd(-13,1) = 1$. It is worth noting that $\gcd(a,0) = a$ if a is positive and $\gcd(a,0) = -a$ if a is negative. (Those of you comfortable with absolute values can condense this to $\gcd(a,0) = |a|$.) Also, $\gcd(b,1) = 1$ for every integer b. Can you see why these formulas are true from the definition?

What happens if the integers are large? For example, suppose we want to calculate $\gcd(1977, 2007)$. Your first instinct might be to try to factor 1977 and 2007 to find their positive divisors, then compare lists to find the largest of all common divisors. Let's try this.

First, $1977 = 3 \times 659$. After a bit of painful trial and error, we find that 659 appears to be a prime number. (How do we know that 659 is prime? That's a subject for another Paragon!) This tells us that the positive divisors of 1977 are 1, 3, 659, and 1977.

Next, $2007 = 3 \times 669 = 3 \times 3 \times 223$. Again, after a bit of flailing around, we discover that 223 is prime; hence, the positive divisors of 2007 are 1, 3, 9, 223, 669, and 2007.

Therefore, the positive common divisors of 1977 and 2007 are 1 and 3, which implies that $\gcd(1977, 2007) = 3$.

This method works reasonably well, but it would be pretty gruesome if 1977 and 2007 were each eight digits long instead of only four, or if there were no readily apparent small prime factors. We can cut down our work a bit by looking directly at the prime factorizations instead of listing divisors, but this still requires calculating the prime factorizations, which, as it turns out, is a very demanding problem computationally.

There is a better way, which takes advantage of the following fact:

> *Important Fact #1:* If a, b, q, and r are integers with $a = qb + r$, then $\gcd(a, b) = \gcd(b, r)$.

This fact is not all that intuitive (lots of number theory books contain a proof, if you're interested), but we can use this fact to do what mathematicians love to do: take a problem and turn it into a smaller (or simpler) one. Here's how:

$$\begin{array}{rcllrcl} 2007 & = & 1(1977) + 30 & \Longrightarrow & \gcd(2007, 1977) & = & \gcd(1977, 30) \\ 1977 & = & 65(30) + 27 & \Longrightarrow & \gcd(1977, 30) & = & \gcd(30, 27) \\ 30 & = & 1(27) + 3 & \Longrightarrow & \gcd(30, 27) & = & \gcd(27, 3) \\ 27 & = & 9(3) + 0 & \Longrightarrow & \gcd(27, 3) & = & \gcd(3, 0) \end{array}$$

Following through this chain, $\gcd(2007, 1977) = \gcd(3, 0)$, which equals 3. (We could have stopped earlier when we saw a gcd that was easy to calculate, but it doesn't hurt to keep going until we get a 0.) Can you tell what we did at each step? At each step, we took out as many copies of the smaller number as we could from the larger number, and determined what was left over. (In technical terms, we performed the Division Algorithm several times, calculating the remainder at each stage.) Overall, this method of calculating the gcd is called the Euclidean Algorithm. Try this algorithm on 6540 and 1236. (Did you get 12 as your answer?)

After you get comfortable with this method, you may notice two time-saving features. The first is that you don't have to write all of the equalities of gcd's down the right side—these will always be true, so we can relate the gcd of the original numbers to the gcd of the final numbers directly. The second builds on the first—the gcd will actually always be the final non-zero remainder in the Algorithm. (Can you see why?)

Manipulations. These methods seem to work really well for numbers, you may say, but can I use them in a more abstract setting, like what might appear in a contest problem?

Funny you should ask .... Here is the very first problem from the very first International Mathematical Olympiad in 1959:

> *Problem #1.* Prove that the fraction $\dfrac{21n + 4}{14n + 3}$ is irreducible for every natural number n.

Step one here, as in any problem, is to figure out what it is really asking. This problem can be restated as "Prove that $\gcd(21n + 4, 14n + 3) = 1$ for every natural number n" (since a fraction is irreducible if its numerator and denominator have no common factors).

We try to model our method from above:

$$\begin{aligned} 21n+4 &= 1(14n+3)+(7n+1)\,, \\ 14n+3 &= 2(7n+1)+1\,, \\ 7n+1 &= (7n+1)(1)+0\,. \end{aligned}$$

Thus, $\gcd(21n+4, 14n+3) = \gcd(14n+3, 7n+1) = \gcd(7n+1, 1) = 1$, as we wanted. So, we can adapt this method!

Another fact that can be quite handy:

Important Fact #2: If $\gcd(c, b) = 1$, then $\gcd(ac, b) = \gcd(a, b)$.

This fact is actually useful in both directions—it allows us to convert $\gcd(a, b)$ to $\gcd(ac, b)$ (although it is not immediately obvious why we would ever want to do this), and it allows us to convert $\gcd(ac, b)$ to $\gcd(a, b)$. This fact is more intuitive—can you explain it to yourself?

We now try a second problem:

Problem #2. Prove that $\gcd(n^2, 2n+1) = 1$ for any natural number n.

Our initial instinct is to try to use the abstract version of the Euclidean Algorithm, but it is very difficult to make $2n+1$ go into n^2 without introducing fractions. This is where Important Fact #2 can be used: since $2n+1$ is odd, then $\gcd(2n+1, 2) = 1$. Thus,

$$\begin{aligned} \gcd(n^2, 2n+1) &= \gcd(2n^2, 2n+1) && (\text{since } \gcd(2n+1, 2) = 1) \\ &= \gcd(-n, 2n+1) && (\text{since } 2n^2 = n(2n+1) + (-n)) \\ &= \gcd(n, 2n+1) && (\text{since } \gcd(-1, 2n+1) = 1) \\ &= \gcd(n, 1) && (\text{since } 2n+1 = 2(n)+1) \\ &= 1\,, \end{aligned}$$

as required.

I hope you have remembered a bit and learned a bit about gcd's here. By no means is what we have done comprehensive, but it should give you some ideas to think about and some strategies to use. Try applying them to one of this month's Mayhem problems!

Ian VanderBurgh
Centre for Education in Mathematics and Computing
University of Waterloo
200 University Avenue West
Waterloo, ON, Canada N2L 3G1
iwtvande@uwaterloo.ca

THE OLYMPIAD CORNER

No. 263

R.E. Woodrow

We begin this number of the *Corner* with selected problems from the Thai Mathematical Olympiad 2003. Thanks go to Christopher Small, Canadian Team Leader to the IMO in Athens, for collecting them for us.

THAI MATHEMATICAL OLYMPIAD 2003
Selected Problems

1. Triangle ABC has $\angle A = 70^\circ$ and $CA + AI = BC$, where I is the incentre of triangle ABC. Find $\angle B$.

2. Let $f : \mathbb{Q} \to \mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers, such that

$$f(x+y) \;=\; f(x) + f(y) + 2547$$

for all x, $y \in \mathbb{Q}$ and $f(2004) = 2547$. Find $f(2547)$.

3. Let a, b, and c be positive real numbers such that $a+b+c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Prove that $a^3 + b^3 + c^3 \geq a + b + c$.

4. Let ABC be an equilateral triangle. Let A', B', and C' be points on the segments BC, CA, and AB, respectively. Suppose that $|AC'| = 2|CB'|$, $|BA'| = 2|AC'|$, $|CB'| = 2|BA|$, and $[ABC] = 126$. Find the area of the triangle enclosed by the lines AA', BB', and CC'.

5. Find all pairs (x, y) which satisfy the system of equations

$$\begin{aligned} x^{x+y} &= y^{xy}\,, \\ x^2 y &= 1\,. \end{aligned}$$

6. Let $ABCD$ be a convex quadrilateral. Prove that

$$[ABCD] \;\leq\; \tfrac{1}{4}\left(AB^2 + BC^2 + CD^2 + DA^2\right).$$

7. Define f on the set of rational numbers in the interval $[0,1]$ as follows: $f(0) = 0$, $f(1) = 1$, and

$$f(x) \;=\; \begin{cases} \dfrac{f(2x)}{4} & \text{if } 0 < x < \dfrac{1}{2}\,, \\ \dfrac{3}{4} + \dfrac{f(2x-1)}{4} & \text{if } \frac{1}{2} \leq x < 1\,. \end{cases}$$

If we write x in base-2 representation as $x = (0.b_1b_2b_3\ldots)_2$, find $f(x)$ in base-2 representation.

8. Find all primes p such that $p^2 + 2543$ has less than 16 distinct positive divisors.

9. Given a right triangle ABC with $\angle B = 90^\circ$, let P be a point on the angle bisector of $\angle A$ inside ABC and let M be a point on the side $\overline{AB}$ (with $A \neq M \neq B$). Lines AP, CP, and MP intersect $\overline{BC}$, $\overline{AB}$, and $\overline{AC}$ at D, E, and N, respectively. Suppose that $\angle MPB = \angle PCN$ and $\angle NPC = \angle MBP$. Find $[APC]/[ACDE]$.

Next we look at two tests of the 25th Albanian Mathematical Olympiad for High Schools. Thanks again go to Christopher Small, Canadian Team Leader to the IMO in Athens, for collecting them.

25th ALBANIAN MATHEMATICAL OLYMPIAD FOR HIGH SCHOOLS

Test 1

1. There are 20 pupils in a village school. Any two of them have the same grandfather. Show that there exists a grandfather who has at least 14 grandchildren.

2. Let M, N, and P be the respective mid-points of sides BC, CA, and AB of triangle ABC, and let G be the intersection point of its medians. Prove that if $BN = \frac{\sqrt{3}}{2}AB$ and $BMGP$ is a cyclic polygon, then triangle ABC is equilateral.

3. Let x_k and y_k (for $k = 1, 2, \ldots, n$) be positive real numbers that satisfy $kx_ky_k \geq 1$.

(a) Prove that $\sum\limits_{k=1}^{n} \dfrac{x_k - y_k}{x_k^2 + y_k^2} \leq \frac{1}{4}n\sqrt{n+1}$.

(b) When does equality hold in part (a)?

4. Find prime numbers p and q such that $p^2 - p + 1 = q^3$.

5. Find all pairs of positive integers (x, n) such that $x^{n+1} + 2^{n+1} + 1$ is divisible by $x^n + 2^n + 1$.

Test 2

1. Some people take part in a meeting. Every participant is acquainted with at most three people in the group, and if two participants are not acquainted, then they have a common acquaintance in the group.

(a) What is the maximal number of participants in this meeting?

(b) If there are three participants who are mutually acquainted with each other, what is the maximal number of participants in this meeting?

2. Prove the inequality

$$\frac{1}{\sqrt{a+\frac{1}{b}+0.64}}+\frac{1}{\sqrt{b+\frac{1}{c}+0.64}}+\frac{1}{\sqrt{c+\frac{1}{a}+0.64}} \geq 1.2\,,$$

where $a > 0$, $b > 0$, $c > 0$, and $abc = 1$.

3. Solve the following equation in integers:

$$y^2 \;=\; 1+x+x^2+x^3+x^4\,.$$

4. Prove that for any integer $n \geq 2$, the number $2^n - 1$ is not divisible by n.

5. In an acute-angled triangle ABC, let H be the orthocentre, and let d_a, d_b, and d_c be the distances from H to the sides BC, CA, and AB, respectively. Prove that $d_a + d_b + d_c \leq 3r$, where r is the radius of the incircle of triangle ABC.

To continue your return to problem-solving pleasures, we give the 11^{th} Form of the Final Round of the 44^{th} Ukrainian Mathematical Olympiad. Thanks again go to Christopher Small for collecting them for our use.

44^{th} UKRAINIAN MATHEMATICAL OLYMPIAD
11^{th} Form, Final Round

1. (V.M. Leifura) Solve the equation

$$\arcsin\lfloor\sin x\rfloor \;=\; \arccos\lfloor\cos x\rfloor\,,$$

where $\lfloor a\rfloor$ is the greatest integer not exceeding a.

2. (V.V. Lymanskiy) The acute-angled triangle ABC is given. Let O be the centre of its circumcircle. The perpendicular bisector of the side AC intersects the side AB and the line BC at the points P and Q, respectively. Prove that $\angle PQB = \angle PBO$.

3. (V.A. Yasinskiy) The edge SA of the tetrahedron $SABC$ is perpendicular to the plane ABC. Two different spheres σ_1 and σ_2 contain points A, B, and C. Both these spheres are tangent internally to a sphere σ centred at S. Let r_1 and r_2 be the radii of σ_1 and σ_2, respectively. Find the radius R of σ.

4. (V.A. Yasinskiy) Prove that there does not exist an integer $n > 1$ such that n divides $3^n - 2^n$.

5. (V.A. Yasinskiy) Given are 2004 points in the plane. They are the vertices of a convex polygon and no four of them are cyclic. A triangle having three of the points as its vertices is called *thick* if the other 2001 points lie inside its circumcircle, and it is called *thin* if the other points lie outside its circumcircle. Prove that the number of *thick* triangles is equal to the number of *thin* triangles.

6. (O.O. Malakhov) Find the sum of the real roots of the equation

$$x + \frac{x}{\sqrt{x^2 - 1}} = 2004\,.$$

7. (V.M. Radchenko) Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f\left(x^2y + f(x + y^2)\right) = x^3 + y^3 + f(xy)$ for all x, $y \in \mathbb{R}$.

8. (V.A. Yasinskiy) Let a, b, and c be positive real numbers such that $abc \geq 1$. Prove that $a^3 + b^3 + c^3 \geq ab + bc + ca$.

9. (V.A. Yasinskiy) A convex 2004-gon has vertices A_1, A_2, $\ldots$, A_{2004}. Is it possible to colour each of its sides and its diagonals with one of 2003 different colours in such a way that the following two conditions hold?

(i) There are 1002 segments of each colour.

(ii) If an arbitrary vertex and two arbitrary colours are given, then, starting from this vertex and using the segments of these two colours exclusively, one can visit every other vertex only once.

10. (I.P. Nagel) Let ω be the inscribed circle of the triangle ABC. Let L, N, and E be the points of tangency of ω with the sides AB, BC, and CA, respectively. Lines LE and BC intersect at the point H, and lines LN and AC intersect at the point J (all the points H, J, N, E lie on the same side of the line AB). Let O and P be the mid-points of the segments EJ and NH, respectively. Find $S(HJNE)$ if $S(ABOP) = u^2$ and $S(COP) = v^2$. (Here $S(\mathcal{F})$ is the area of figure $\mathcal{F}$).

We turn to our files of readers' comments and solutions to problems given in the September 2006 number of the *Corner*. The first group are for problems of the Belarus Mathematical Olympiad 2003, given in [2006 : 277].

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

These six problems are from the IMO Short-list 2001.
Solutions can be found at

`http://www.mathlinks.ro/Forum/viewtopic.php?t-15624`

or in D Djukić, V. Janković, I. Matić, N. Petrović, *The IMO Compendium*, Springer, p.675.

3. Find all functions f from the real numbers to the real numbers such that, for any real numbers x and y,

$$f(xy)\big(f(x)-f(y)\big) \;=\; (x-y)f(x)f(y)\,.$$

Solved by Michel Bataille, Rouen, France.

Clearly, the zero function is a solution.

We will show that the non-zero solutions are the functions $\Phi_{a,K}$ defined by

$$\Phi_{a,K}(x) \;=\; \begin{cases} ax & \text{if } x \in K, \\ 0 & \text{if } x \notin K, \end{cases}$$

where K is a subgroup of the multiplicative group $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$ and $a \in \mathbb{R}^*$.

Consider $a \in \mathbb{R}^*$ and a subgroup K of $\mathbb{R}^*$. We show that $f = \Phi_{a,K}$ satisfies

$$f(xy)\big(f(x)-f(y)\big) \;=\; (x-y)f(x)f(y) \tag{1}$$

for all x, $y \in \mathbb{R}$.

If x, $y \in K$, then $xy \in K$ and (1) holds since it can be rewritten as $axy(ax-ay) = (x-y)ax\cdot ay$.

If x, $y \notin K$, then $\Phi(x) = \Phi(y) = 0$ and (1) is true.

If, say, $x \in K$, $y \notin K$, certainly $xy \notin K$ (otherwise $y = xy\cdot\dfrac{1}{x}$ would be in K). Thus, $\Phi(xy) = \Phi(y) = 0$ and (1) holds.

Conversely, let f be any function from $\mathbb{R}$ to $\mathbb{R}$ satisfying (1), and assume that f is not the zero function. Then $f(x_0) \neq 0$ for some $x_0 \in \mathbb{R}$. Taking $x = 1$ and $y = 0$ in (1) yields $f(0) = 0$ (hence, $x_0 \neq 0$). Then $y = 1$ gives

$$f(x)\big(f(x)-ax\big) \;=\; 0\,, \tag{2}$$

where we set $a = f(1)$. This relation (2) with $x = x_0$ shows that $a \in \mathbb{R}^*$ and, more generally, that $f(x) = ax$ if $f(x) \neq 0$.

Now, let

$$K \;=\; \{x \in \mathbb{R}^* \mid f(x) = ax\} \;=\; \{x \in \mathbb{R} \mid f(x) \neq 0\}\,.$$

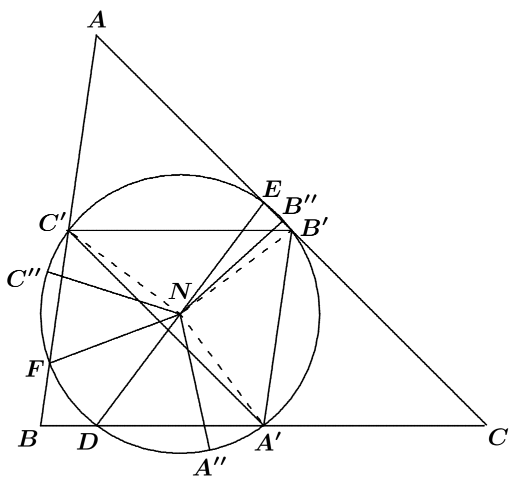
Clearly $1 \in K$. If x_1, $x_2 \in K$ with $x_1 \neq x_2$, then $f(x_1) = ax_1$ and $f(x_2) = ax_2$. Then (1) with $x = x_1$ and $y = x_2$ gives $f(x_1x_2) = ax_1x_2$; hence, $x_1x_2 \in K$. Similarly, (1) with $x = x_1$ and $y = 1/x_1$ shows that $1/x_1 \in K$ and, lastly, (1) with $x = x_1^2$ and $y = 1/x_1$ gives $x_1^2 \in K$. We have proved that K is a subgroup of $\mathbb{R}^*$ and that $f = \Phi_{a,K}$. This completes the proof.

The next block of material is for the Problems to Select Indian IMO Team 2003 given in [2006 : 278–279].

1. Let A', B', C' be the mid-points of the sides BC, CA, AB, respectively, of an acute non-isoceles triangle ABC, and let D, E, F be the feet of the altitudes through the vertices A, B, C on these sides, respectively. Consider the arc DA' of the nine-point circle of triangle ABC lying outside the triangle. Let the point of trisection of this arc closer to A' be A''. Define analogously the points B'' (on arc EB') and C'' (on arc FC'). Show that triangle $A''B''C''$ is equilateral.

Solved by Michel Bataille, Rouen, France; and D.J. Smeenk, Zaltbommel, the Netherlands. First we give Bataille's solution.

Let N be the centre of the nine-point circle $\mathcal{N}$ of triangle ABC. The perpendicular bisectors of $B'C'$ and $A'D$ coincide (both pass through N and are perpendicular to BC); hence, A' and D are symmetrical in the diameter of $\mathcal{N}$ perpendicular to BC. It follows that the mid-point of the (smaller) arc $B'C'$ and the mid-point of the arc DA' lying outside the triangle are diametrically opposite.



Without loss of generality, we may suppose that $\mathcal{N}$ is the unit circle in the complex plane, with $\triangle A'B'C'$ positively oriented (see figure). We may even suppose that the angles $\alpha = \angle B'A'C'$, $\beta = \angle C'B'A'$, and $\gamma = \angle A'C'B'$ satisfy $\frac{\pi}{2} > \beta > \alpha > \gamma$ and that the complex affix of B' is 1. It is readily seen that the affixes of C' and A' are $e^{2i\alpha}$ and $e^{-2i\gamma} = e^{2i(\pi-\gamma)}$, respectively. The affix of the mid-point of the smaller arc $B'C'$ is $e^{i\alpha}$; hence, the affix of the mid-point of the arc DA' is $e^{i(\pi+\alpha)}$. Since $\pi + \alpha < 2\pi - 2\gamma$ (note that $\alpha + 2\gamma < \alpha + \beta + \gamma - \pi$), we find D, A'', and A' in that order on the circle positively oriented. It follows that the affix of A'' is

$$e^{i(\pi+\alpha+(2\pi-2\gamma-\pi-\alpha)/3)} = e^{4\pi i/3} \cdot e^{2i(\alpha-\gamma)/3} .$$

In a similar way, we find that the affix of B'' is $e^{2i(\alpha-\gamma)/3}$ and the affix of C'' is $e^{2\pi i/3}\cdot e^{2i(\alpha-\gamma)/3}$. Thus, the affixes of B'', C'', and A'' are of the form $e^{i\theta}$, $e^{i\theta}\cdot e^{2\pi i/3}$, and $e^{i\theta}\cdot e^{4\pi i/3}$; whence, $\triangle A''B''C''$ is equilateral.

Next we give Smeenk's approach.

Set $a = BC$, $b = CA$, $c = AB$, $\alpha = \angle CAB$, $\beta = \angle ABC$, and $\gamma = \angle BCA$. Without loss of generality, we may assume that $\beta > \alpha > \gamma$. We first note that $BA' = \frac{1}{2}a$ and $BD = c\cos\beta$. Then

$$DA' \;=\; BA' - BD \;=\; \tfrac{1}{2}a - c\cos\beta \;=\; R\sin(\beta-\gamma)\,,$$

where R is the circumradius of $\triangle ABC$. Similarly, $EB' = R\sin(\alpha-\gamma)$.

Let N be the centre of the nine-point circle of $\triangle ABC$. Then we have $\angle DNA' = 2(\beta-\gamma)$ and $\angle ENB' = 2(\alpha-\gamma)$. In $\triangle NA'B'$ we have $\angle A'NB' = 2\gamma$. Therefore,

$$\angle A''NB'' \;=\; 2\gamma + \tfrac{2}{3}(\beta-\gamma) + \tfrac{2}{3}(\alpha-\gamma) \;=\; \tfrac{2}{3}(\alpha+\beta+\gamma) \;=\; 120^\circ\,.$$

In the same way, $\angle B''NC'' = \angle C''NA'' = 120^\circ$. Thus, $\triangle A''B''C''$ is equilateral.

2. Find all triples (a,b,c) of positive integers such that

(i) $a \le b \le c$;

(ii) $\gcd(a,b,c) = 1$; and

(iii) $a^3+b^3+c^3$ is divisible by each of the numbers a^2b, b^2c, c^2a.

Solution par Pierre Bornsztein, Maisons-Laffitte, France.

Soit (a,b,c) un tel triplet.

On remarque d'abord que si p premier divise a et b, il divise a^2b et donc $a^3+b^3+c^3$ ainsi que a^3+b^3. Par suite, p divise c^3 et donc c, ce qui contredit (ii). Ainsi, $\gcd(a,b)=1$. De même, $\gcd(b,c)=\gcd(c,a)=1$.

On en déduit que a^2, b^2 et c^2 sont deux à deux premiers entre eux. Puisqu'ils divisent chacun $a^3+b^3+c^3$, cela assure que $a^2b^2c^2$ divise $a^3+b^3+c^3$. On a alors $3c^3 \ge a^3+b^3+c^3 \ge a^2b^2c^2$, d'où

$$3c \;\ge\; a^2b^2\,. \tag{1}$$

D'autre part, c^2 divise a^3+b^3, donc

$$c^2 \;\le\; a^3+b^3 \;\le\; 2b^3\,. \tag{2}$$

En combinant (1) et (2), il vient $108c^3 \ge 4a^6b^6 \ge a^6c^4$ et donc $a \le c \le 108/a^6$, et enfin $a^7 \le 108$. Cela entraine $a=1$.

Si $b=1$, alors c^2 divise $a^3+b^3=2$, d'où $c=1$. Réciproquement, le triplet $(1,1,1)$ est clairement une solution.

Si $b>1$, alors $c>b>1$ (sans quoi, on aurait $c=b$, en contradiction avec $\gcd(b,c)=1$). Et donc $c^3 > b^3+1$. Il vient alors :

$$2c^3 \;>\; 1+b^3+c^3 \;=\; a^3+b^3+c^3 \;\ge\; b^2c^2\,,$$

d'où $c > \frac{1}{2}b^2$. En combinant avec (2), il vient $2b^3 > \frac{1}{4}b^4$, ou encore $b < 8$. On vérifie alors à la main que la seule solution est $b = 2$ et $c = 3$.

Finalement, les solutions sont $(1, 1, 1)$ et $(1, 2, 3)$.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all x, y in $\mathbb{R}$, we have

$$f(x+y) + f(x)f(y) \;=\; f(x) + f(y) + f(xy)\,. \tag{1}$$

Solved by Michel Bataille, Rouen, France; and Li Zhou, Polk Community College, Winter Haven, FL, USA. We give the write-up by Zhou, modified by the editor.

Clearly, $f \equiv 0$, $f \equiv 2$, and $f(x) = x$ are solutions. We show that they are the only solutions.

Setting $x = 0 = y$ in (1), we get $\big(f(0)\big)^2 = 2f(0)$. If $f(0) = 2$, then, by letting $y = 0$ in (1), we see that $f(x) = 2$ for all $x \in \mathbb{R}$; that is, $f \equiv 2$.

Suppose therefore that $f(0) = 0$. Let $a = f(1)$. Setting $x = 1$ and $y = -1$, we get $af(-1) = a + 2f(-1)$; that is, $f(-1) = a/(a-2)$. Now successively substituting $(x-1, 1)$, $(-x+1, -1)$, and $(-x, 1)$ for (x, y) in (1), we get

$$\begin{aligned} f(x) + (a-2)f(x-1) &= a\,, & (2)\\ f(-x) + \frac{2}{a-2}f(-x+1) - f(x-1) &= \frac{a}{a-2}\,, & (3)\\ f(-x+1) + (a-2)f(-x) &= a\,. & (4) \end{aligned}$$

Eliminating $f(x-1)$ and $f(-x+1)$ in (2), (3), and (4) gives

$$f(x) - (a-2)f(-x) \;=\; 0\,. \tag{5}$$

Replacing x by $-x$ in (5), we obtain

$$f(-x) - (a-2)f(x) \;=\; 0\,. \tag{6}$$

If $a \notin \{1,\, 3\}$, then by eliminating $f(-x)$ in (5) and (6), we see that $f(x) = 0$ for all $x \in \mathbb{R}$; that is, $f \equiv 0$.

If $a = 3$, then (2) gives $f(x) = 3 - f(x-1)$ for all $x \in \mathbb{R}$. Hence, $f(2) = 3 - f(1) = 0$ and $f\big(\frac{5}{2}\big) = 3 - f\big(\frac{3}{2}\big) = f\big(\frac{1}{2}\big)$. On the other hand, by substituting $\big(2, \frac{1}{2}\big)$ for (x, y) in (1), we get $f\big(\frac{5}{2}\big) = f\big(\frac{1}{2}\big) + f(1) = f\big(\frac{1}{2}\big) + 3$, a contradiction.

Finally, consider $a = 1$. Then (2) gives $f(x) = f(x-1) + 1$ for all $x \in \mathbb{R}$. By induction, $f(x+n) = f(x) + n$, for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. In particular, $f(n) = f(0+n) = f(0) + n = n$ for all $n \in \mathbb{Z}$. Substituting n for y in (1), we obtain $nf(x) = f(nx)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. Hence, if $r = m/n \in \mathbb{Q}$ and $x \in \mathbb{R}$, then

$$mf(x) \;=\; f(mx) \;=\; f(n \cdot rx) \;=\; nf(rx)\,;$$

that is, $f(rx) = rf(x)$. In particular, $f(r) = f(r \cdot 1) = rf(1) = r$. Setting $y = r$ in (1) gives $f(x+r) = f(x) + r$. Also, setting $y = x$ in (1), we get $\big(f(x)\big)^2 = f(x^2)$. Thus, $f(x) \geq 0$ if $x \geq 0$. Since $f(-x) = -f(x)$, we have $f(x) \leq 0$ if $x \leq 0$.

Now, let $x \in \mathbb{R}$ be fixed. If $r \in \mathbb{Q}$ and $r < x$, then

$$f(x) \;=\; f(x-r+r) \;=\; f(x-r)+r \;\geq\; r\,,$$

since $f(x-r) \geq 0$. It follows that $f(x) \geq x$. Similarly, $f(x) \leq r$ for all $r \in \mathbb{Q}$ such that $r > x$, which implies that $f(x) \leq x$. Thus, $f(x) = x$.

7. Let $P(x)$ be a polynomial with integer coefficients such that $P(n) > n$ for all positive integers n. Suppose that for each positive integer m, there is a term in the sequence $P(1)$, $P\big(P(1)\big)$, $P\big(P\big(P(1)\big)\big)$, $\ldots$ which is divisible by m. Show that $P(x) = x + 1$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Li Zhou, Polk Community College, Winter Haven, FL, USA. We give Zhou's write-up.

We will use the notation $P^{(i)}(1)$ for the i^{th} term in the sequence $P(1)$, $P\big(P(1)\big)$, $\ldots$. The condition $P(n) > n$ implies that $\deg(P) \geq 1$ and the leading coefficient of P is positive.

If $P(x) = x + b$, then $1 + b = P(1) > 1$, which implies that $b \geq 1$. It is easy to see that $P(x) = x + 1$ satisfies all the conditions. If $b \geq 2$, then $P(1) \equiv 1 \pmod{b}$ and, by induction, $P^{(i)}(1) \equiv 1 \pmod{b}$ for all $i \geq 1$.

If $P(x) = 2x + b$, then $2 + b = P(1) > 1$, which implies that $b \geq 0$. If $b = 0$, then, by induction, $P^{(i)}(1) = 2^i$ for all $i \geq 1$.

We consider together all the remaining cases: (i) $P(x) = 2x + b$ with $b \geq 1$; (ii) $P(x) = ax + b$ with $a \geq 3$; and (iii) $\deg(P) \geq 2$. In these cases, there exists $N \in \mathbb{N}$ such that $P(n) > 2n$ for all $n \geq N$. Since $1 < P(1) < P(P(1)) < P(P(P(1))) < \cdots$, there exists $k \in \mathbb{N}$ such that $P^{(k)}(1) \geq N$. Let $r = P^{(k)}(1)$ and $m = P^{(k+1)}(1) - P^{(k)}(1)$. Then $r \geq N$ and $m = P(r) - r > r$. For $1 \leq i \leq k$, we have $1 < P^{(i)}(1) \leq r < m$; thus, m does not divide any $P^{(i)}(1)$ for $1 \leq i \leq k$. Moreover, note that $P^{(k+1)}(1) = m + r \equiv r \pmod{m}$. Assume as an induction hypothesis that $P^{(i)}(1) \equiv r \pmod{m}$ for some $i \geq k+1$. Then

$$\begin{aligned} P^{(i+1)}(1) &= P(P^{(i)}(1)) \;\equiv\; P(r) \;=\; P(P^{(k)}(1)) \\ &= P^{(k+1)}(1) \;\equiv\; r \pmod{m}\,. \end{aligned}$$

Hence, $P^{(i)}(1) \equiv r \pmod{m}$ for all $i \geq k+1$.

8. Let ABC be a triangle, and let r, r_1 ,r_2, r_3 denote its inradius and the exradii opposite the vertices A, B, C, respectively. Suppose $a > r_1$, $b > r_2$, $c > r_3$. Prove that

(a) triangle ABC is acute,

(b) $a + b + c > r + r_1 + r_2 + r_3$.

Solution by Vedula N. Murty, Dover, PA, USA, modified by the editor.

(a) Let s denote the semipermieter of $\triangle ABC$. From the known formula $\tan(A/2) = r_1/s$ and the given inequality $a > r_1$, we obtain $\tan(A/2) < a/s < 1$. Similarly, $\tan(B/2) < 1$ and $\tan(C/2) < 1$. Then $A < \frac{\pi}{2}$, $B < \frac{\pi}{2}$, and $C < \frac{\pi}{2}$. Hence, $\triangle ABC$ is acute.

(b) Since the triangle ABC is acute, we have the known inequality $s > r + 2R$, where R is the circumradius of $\triangle ABC$. We also have $r_1 + r_2 + r_3 = r + 4R$. Hence,

$$r + r_1 + r_2 + r_3 \;=\; 2r + 4R \;<\; 2s \;=\; a + b + c\,.$$

9. Let n be a positive integer and $\{A, B, C\}$ a partition of $\{1, 2, \ldots, 3n\}$ such that $|A| = |B| = |C| = n$. Prove that there exist $x \in A$, $y \in B$, $z \in C$ such that one of x, y, z is the sum of the other two.

Solution par Pierre Bornsztein, Maisons-Laffitte, France.

On dira que le triplet (a, b, c) est *bon* lorsque $a \in A$, $b \in B$, $c \in C$ et l'un des nombres est la somme des deux autres. Sans perte de généralité, on peut supposer que $1 \in A$, et que le plus petit nombre, disons k, qui n'est pas dans A est dans B. Supposons qu'il n'existe pas de bon triplet.

On commence par montrer que pour tout $x \in C$, on a $x - 1 \in A$. En effet, s'il existe $x \in C$ tel que $x - 1 \notin A$ alors, puisque $(1, x - 1, x)$ ne doit pas être bon, c'est donc que $x - 1 \notin B$, et ainsi que $x - 1 \in C$. En particulier, $x - 1 > k$. Mais comme aucun des deux triplets $(x - k, k, x)$ et $(k - 1, x - k, x - 1)$ n'est bon, c'est que $x - k \notin A$ et $x - k \notin B$. Et donc $x - k \in C$. De même, en considérant les triplets $(x - k - 1, k, x - 1)$ et $(1, x - k - 1, x - k)$, on déduit que $x - k - 1 \in C$.

On peut alors recommencer ce raisonnement, et prouver par récurrence que, pour tout $i \geq 0$, les nombres $x - ik$ et $x - ik - 1$, tant qu'ils sont strictement positifs, appartiennent tous les deux à C. Mais pour un i bien choisi, un de ces nombres est nécessairement inférieur ou égal à k, ce qui implique qu'il appartienne à A ou à B. C'est une contradiction.

On pose $C = \{c_1, \ldots, c_n\}$. D'après la propriété précédente, et puisque $|A| = |C| = n$, c'est donc que $A = \{c_1 - 1, \ldots, c_n - 1\}$. Mais pour tout i, on a $c_i > k > 1$ donc $c_i - 1 > 1$, ce qui contredit que $1 \in A$.

Remarque. On peut prouver que si A, B et C forment une partition de $\{1, 2, \ldots, n\}$ avec $|A|$, $|B|$, $|C| > \frac{1}{4}n$, alors il existe un bon triplet.

Référence.

[1] G.J. Székely, *Contests in higher mathematics*, Springer, problem C-22.

10. Let n be a positive integer greater than 1, and let p be a prime such that n divides $p - 1$ and p divides $n^3 - 1$. Prove that $4p - 3$ is a square.

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

This problem is similar to problem #4 of the 3rd Czeck-Polish-Slovak mathematical competition. A solution appears in [2006 : 375–376].

Next we turn to readers' solutions to problems of the German Mathematical Olympiad 2003 given at [2006 : 279–280].

1. Determine all pairs (x, y) of real numbers x, y which satisfy

$$\begin{aligned} x^3 + y^3 &= 7\,, \\ xy(x+y) &= -2\,. \end{aligned}$$

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bornsztein's write-up.

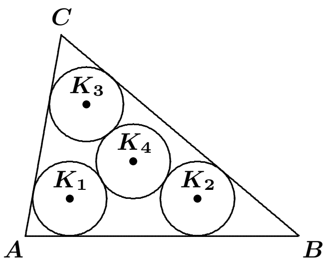
Assume that (x, y) is such a pair. Then

$$(x+y)^3 \;=\; x^3 + y^3 + 3xy(x+y) \;=\; 1\,,$$

which leads to $x + y = 1$. Thus, $xy = -2$. It follows that x and y are roots of $X^2 - X - 2 = 0$. Therefore, $(x, y) = (-1, 2)$ or $(2, -1)$, which are solutions of the problem.

2. In the interior of a triangle ABC, circles K_1, K_2, K_3, and K_4 of the same radii are defined such that K_1, K_2, and K_3 touch two sides of the triangle and K_4 touches K_1, K_2, and K_3, as shown in the figure.

Prove that the centre of K_4 is located on the line through the incentre and the circumcentre.



Solved by Michel Bataille, Rouen, France; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Bataille's solution.

Let O_i be the centre of the circle K_i (for $i = 1, 2, 3, 4$), and let ρ be the common radius of the circles. Let γ (centre I, radius r) and Γ (centre O, radius R) denote the incircle and the circumcircle of $\triangle ABC$, respectively.

Circle K_1 is the image of γ under the homothety with centre A and scale factor $k = \rho/r$; hence, $\overrightarrow{AO_1} = k\overrightarrow{AI}$, so that $\overrightarrow{IO_1} = (1-k)\overrightarrow{IA}$. Similarly, $\overrightarrow{IO_2} = (1-k)\overrightarrow{IB}$ and $\overrightarrow{IO_3} = (1-k)\overrightarrow{IC}$. Therefore, $\triangle O_1O_2O_3$ is the image of $\triangle ABC$ under the homothety with centre I and factor $1-k$. It follows that the circumcentre of $\triangle O_1O_2O_3$, namely O_4, is the image of O through this homothety. As a result, O_4, I, and O are collinear, as required.

Note. The circumradius 2ρ of $\triangle O_1O_2O_3$ satisfies $2\rho = \left(1 - \frac{\rho}{r}\right)R$. Thus, $\rho = \dfrac{rR}{2r+R}$, a useful result when drawing the figure.

3. The caterpillar *Nummersatt* is sitting in the middle square of an $N \times N$ board, where N is an odd integer with $N \geq 3$. The other squares of the board each contain a positive integer, and all of these integers are different. Nummersatt wants to find a way off the board. The caterpillar can move only between adjacent squares (squares having a common side), or off the board from one of the outermost squares, having once reached such a square. On reaching a new square, Nummersatt has to eat the number on that square. The number n weighs $\frac{1}{n}$ kg, and Nummersatt cannot eat more than 2 kg.

Decide whether numbers can be distributed on the board so that there is no way off the board for Nummersatt

(a) for $N = 2003$, (b) for all odd integers $N \geq 3$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France.

Nous allons prouver que, dans le cas général, Nummersatt peut toujours sortir du tableau sans manger plus de $0,9$ kg. Pour cela, on va utiliser la méthode probabiliste.

Tout d'abord, on remarque que, quitte à augmenter le poid avalé, on peut supposer que les entiers utilisés sont 1, 2, $\ldots$, N^2.

On identifie chaque case avec son centre de sorte que, si $N = 2k + 1$, chaque case est un point du réseau des points entiers à coordonnées dans $I = \{-k, -k + 1, \ldots, k - 1, k\}$. Le carré central est $(0, 0)$.

On va considérer les chemins partant de $(0, 0)$ et sortant du réseau en ne passant que d'un point (x, y) à un point voisin (x', y') tel que $|x'| + |y'| > |x| + |y|$ (par exemple, si x, $y > 0$ cela n'autorise qu'un déplacement vers $(x + 1, y)$ ou vers $(x, y + 1)$). En particulier, un tel chemin mène nécessairement vers la sortie en un nombre fini d'étapes et sans boucle. Le long d'un tel chemin, Nummersatt se déplace en respectant les conditions de l'énoncé, celle indiquée ci-dessus et les contraintes probabilistes suivantes (il est vivement conseillé de faire un dessin) :

- De $(0, 0)$, il choisit un des points $(-1, 0)$, $(1, 0)$, $(0, 1)$, and $(0, -1)$ de façon équiprobable.
- S'il est en $(n, 0)$, avec $n \geq 1$, il va en $(n, 1)$ ou en $(n, -1)$ avec une probabilité de $\dfrac{1}{2(n+1)}$ dans chacun des cas, et va en $(n + 1, 0)$ avec une probabilité de $\dfrac{n}{n+1}$.
- S'il est en $(n - p, p)$, avec $n > p \geq 1$, il va en $(n - p + 1, p)$ avec une probabilité de $\dfrac{2(n-p)+1}{2(n+1)}$ et en $(n - p, p + 1)$ avec une probabilité de $\dfrac{2p+1}{2(n+1)}$.
- S'il est en $(0, n)$, avec $n \geq 1$, il va en $(1, n)$ ou en $(-1, n)$ avec une probabilité de $\dfrac{1}{2(n+1)}$ dans chacun des cas, et va en $(0, n + 1)$ avec une probabilité de $\dfrac{n}{n+1}$.

Les autres cas se traitent de façon symétrique (par rapport à l'un des axes de coordonnées ou par rapport à $(0,0)$) par rapport à la situation qui vient d'être décrite pour les points (x,y) avec $x,\ y \geq 0$. Il est alors facile de vérifier que si l'on note $p(x,y)$ la probabilité que Nummersatt passe par le point $(x,y) \neq (0,0)$, alors $p(x,y) = \dfrac{1}{4(|x|+|y|)}$.

Soit X la variable aléatoire égale au poid total mangé par Nummersatt au cours de son périple. On note $E(X)$ son espérance mathématique.

Pour $(x,y) \neq (0,0)$, on note $w(x,y)$ le poids attribué au point (x,y) (si le nombre accroché à (x,y) est n, on a donc $w(x,y) = 1/n$. De plus, on convient, pour simplifier les sommations qui suivent, que $w(x,y) = 0$ lorsque (x,y) ne fait pas partie du réseau fini considéré). On note que pour un point (x,y) du réseau considéré, on a $|x|+|y| \leq 2k$.

Alors :

$$\begin{aligned}
E(X) &= \sum_{(x,y)\neq(0,0)} p(x,y)w(x,y) \\
&= \sum_{n=1}^{2k}\left[\sum_{|x|+|y|=n} p(x,y)w(x,y)\right] = \sum_{n=1}^{2k}\left[\frac{1}{4n}\sum_{|x|+|y|=n} w(x,y)\right] \\
&\leq \frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\frac{1}{8}\left(\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{12}\right) \\
&\qquad +\frac{1}{12}\left(\frac{1}{13}+\cdots+\frac{1}{24}\right)+\cdots \\
&= \frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\frac{1}{8}\left(\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{12}\right)+R,
\end{aligned}$$

où

$$R = \sum_{n=3}^{2k}\left[\frac{1}{4n}\sum_{i=1}^{4n}\frac{1}{2n^2-2n+i}\right].$$

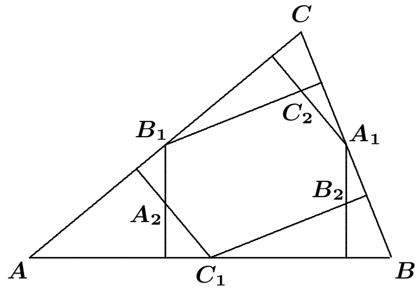
On a $\frac{1}{4}(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4})+\frac{1}{8}(\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{12}) = \frac{143771}{221760} < 0,65$ et

$$\begin{aligned}
R &\leq \sum_{n=3}^{2k}\left[\frac{1}{4n}\sum_{i=1}^{4n}\frac{1}{2n^2-2n}\right] = \sum_{n=3}^{2k}\left(\frac{1}{4n}\right)\left(\frac{4n}{2n^2-2n}\right) \\
&= \sum_{n=3}^{2k}\frac{1}{2n(n-1)} = \frac{1}{2}\sum_{n=3}^{2k}\left(\frac{1}{n-1}-\frac{1}{n}\right) = \frac{1}{2}\left(\frac{1}{2}-\frac{1}{2k}\right) < \frac{1}{4}.
\end{aligned}$$

Ainsi $E(X) < 0,9$.

On en déduit que, parmi les chemins considérés, il en existe un pour lequel le poids total mangé par Nummersatt ne dépasse pas $0,9$ kg.

4. Let A_1, B_1, and C_1 be the mid-points of the sides of the acute-angled triangle ABC. The 6 lines through these points perpendicular to the other sides meet in the points A_2, B_2, and C_2, as shown in the figure. Prove that the area of the hexagon $A_1C_2B_1A_2C_1B_2$ equals half of the area of $\triangle ABC$.



Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Bataille's write-up.

Denote the area of $\triangle XYZ$ by $[XYZ]$. Since $\triangle A_1B_1C_1$ is the image of $\triangle ABC$ under the homothety with centre at the centroid and factor $-\frac{1}{2}$, we have $[A_1B_1C_1] = \frac{1}{4}[ABC]$. Similarly, using the homothety h with centre A and factor $\frac{1}{2}$, we have $[AB_1C_1] = \frac{1}{4}[ABC]$.

Now, let O and H be the circumcentre and orthocentre of $\triangle ABC$, respectively. Note that these points are interior to the acute-angled $\triangle ABC$. Letting h_a be the length of the altitude from A to BC in $\triangle ABC$, we have

$$[BHC] = \tfrac{1}{2}\cdot BC\cdot(h_a - AH) = [ABC] - \tfrac{1}{2}BC\cdot AH\,.$$

Using the well-known relation $AH = 2OA_1$, we deduce that

$$[BHC] = [ABC] - BC\cdot OA_1 = [ABC] - 2[OBC]\,.$$

Observing that $h(H) = A_2$ (where h is the homothety defined above), we obtain

$$[B_1A_2C_1] = \tfrac{1}{4}[BHC] = \tfrac{1}{4}[ABC] - \tfrac{1}{2}[OBC]\,.$$

In the same way, we can show that $[C_1B_2A_1] = \frac{1}{4}[ABC] - \frac{1}{2}[OCA]$ and $[A_1C_2B_1] = \frac{1}{4}[ABC] - \frac{1}{2}[OAB]$. It follows that

$$\begin{aligned}[A_1C_2B_1A_2C_1B_2] &= [A_1B_1C_1] + [B_1A_2C_1] + [C_1B_2A_1] + [A_1C_2B_1]\\ &= \tfrac{1}{4}[ABC] + \tfrac{3}{4}[ABC]\\ &\qquad - \tfrac{1}{2}([OBC] + [OCA] + [OAB])\\ &= [ABC] - \tfrac{1}{2}[ABC] = \tfrac{1}{2}[ABC]\,,\end{aligned}$$

as required.

5. If n is a positive integer, let $a(n)$ be the smallest positive number for which $(a(n))!$ is divisible by n. Determine all positive integers n satisfying

$$\frac{a(n)}{n} = \frac{2}{3}\,.$$

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up.

The only solution is $n = 9$. Note first that the given condition is simply $3a(n) = 2n$, which implies $3 \mid n$. Setting $n = 3k$, we have $a(3k) = 2k$. Clearly, $k \neq 1$ since $a(3) = 3$. For $k \geq 2$, we have $3k \mid (2k)!$, since $(2k)! = (2k)\cdots 2 \cdot 1$ contains the separate factors k and 3 even if $k = 3$. However, if $k > 3$, then $(2k-1)! = (2k-1)\cdots k \cdots 2 \cdot 1$ also contains the distinct factors k and 3, and hence, $a(3k) < 2k$. Finally, for $k = 3$, we have $a(9) = 6$, since $9 \mid 6!$ but $9 \nmid 5!$. Thus, $n = 9$ is the only solution.

6. Prove that there are infinitely many pairs (a, b) of positive integers with $a > b$ having the following properties:

(i) the greatest common divisor of a and b equals 1;

(ii) a is a divisor of $b^2 - 5$.

(iii) b is a divisor of $a^2 - 5$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's write-up.

Let (a, b) be a pair of positive integers with $a > b$. We first show that (a, b) satisfies (i), (ii), and (iii) if and only if $a^2 + b^2 - 5$ is a multiple of ab.

Suppose that (i), (ii), and (iii) hold. Then $b^2 - 5 = \lambda a$ and $a^2 - 5 = \mu b$ for some integers λ and μ. Hence, $ab^2 - 5a = \lambda a^2 = 5\lambda + \lambda\mu b$. Thus, $b(ab - \lambda\mu) = 5(\lambda + a)$. If 5 divides b, then, since $a^2 = 5 + \mu b$, it follows that 5 divides a, contradicting (i). Therefore, 5 divides $ab - \lambda\mu$. Then $ab - \lambda\mu = 5k$ for some integer k, and we have $b(5k) = 5(\lambda + a)$; that is, $\lambda = bk - a$. Then $b^2 - 5 = (bk - a)a$; that is, $a^2 + b^2 - 5 = kab$.

Conversely, suppose that (a, b) satisfies $a^2 + b^2 - 5 = kab$ for some integer k. Conditions (ii) and (iii) clearly hold. If $d = \gcd(a, b)$, then $a = da'$ and $b = db'$ and so $d^2a'^2 + d^2b'^2 - 5 = kd^2a'b'$. It follows that d^2 divides 5 and d must be 1, implying that condition (i) holds as well.

Now, suppose that (a, b) satisfies $a^2 + b^2 - 5 = kab$ for some integer $k > 1$. From $(ak - b)^2 + a^2 - 5 = ka(ak - b)$, we see that $(ak - b, a)$ is another pair satisfying the conditions. Starting with the pair $(4, 1)$ (for which $k = 3$), and applying repeatedly the transformation $(a, b) \to (ak - b, a)$, we obtain a sequence of distinct pairs that are solutions. Specifically, let $a_1 = 4$, $b_1 = 1$ and $a_{n+1} = 3a_n - b_n$, $b_{n+1} = a_n$ for all $n \in \mathbb{N}$, then $a_n < a_{n+1}$ for all n and (a_n, b_n) satisfies all the conditions. Besides, it is easily seen that $a_n = L_{2n+1}$ and $b_n = L_{2n-1}$ where $\{L_n\}$ is the Lucas sequence defined by $L_0 = 2$, $L_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$ for $n \in \mathbb{N}$.

That completes this number of the *Corner*. Please send your nice solutions and generalizations promptly since I start using your material about eight months after the problem sets appear in the *Corner*.

BOOK REVIEW

John Grant McLoughlin

King of Infinite Space—Donald Coxeter, the Man Who Saved Geometry
By Siobhan Roberts, House of Anansi Press, Toronto, 2006
ISBN 0-88784-201-1, hardcover, 480 pages, CDN$36.95.
Reviewed by ***Andy Liu****, University of Alberta, Edmonton, AB*

This book is more than just a biography of the late Professor Coxeter. It is a veritable encyclopedia of geometry for the layman. I must state immediately and emphatically that the phrase "for the layman" is not intended to be derogatory, but high praise. Although I, as a professional mathematician, learned a lot by reading this book, the mathematical details are so well handled (sometimes in the Appendices) that I think the book could be read, absorbingly, by the general public. I welcome this book whole-heartedly as a fine, and all-too-rare, example of how to communicate mathematics effectively.

The subtitle of the book, "*the Man Who Saved Geometry*", is rather bewildering to many people, as illustrated by an amusing incident related by the author in the Acknowledgement. The author sets the scene in the Introduction with a painstaking description of the decline of geometry in the 20^{th} century. The battle cry, "Down with Euclid! Death to Triangles", had been sounded loud and clear by the Bourbakists, a most influential group of French mathematicians. One of them, Jean Dieudonné, is cast in the role of Professor Coxeter's antagonist—though I must hastily add that he is treated with all due respect, representing a different set of values with its own justifications.

Professor Coxeter's field of research was definitely unfashionable and set him back in worldly advancements at first. However, he was determined to hold his course (a recurrent theme of the book). In one of the Appendices, the author quotes extensively from a paper by the renowned physicist, Freeman Dyson of Princeton's Institute of Advanced Studies, on "Unfashionable Pursuits". Dyson once sent a copy to Professor Coxeter, who most certainly appreciated it very much.

The first chapter deals with the last conference Professor Coxeter attended, in Budapest, Hungary. He was accompanied at the conference by family and colleagues, and by the author, who must have been in the process of getting to know the great man. Many first-hand experiences are recorded, with colourful details. Professor Coxeter must have reflected on his illustrious career and discussed with the author his love of geometry. We learn about the accomplishments of the icons of ancient Greece—Pythagoras, Plato, and Euclid—as well as the lives of Hungarian giants of the more recent past, János Bolyai and Paul Erdős.

The remaining seven chapters of Part I relate events in Professor Coxeter's life in chronological order, starting with his childhood in England. For me, the highlight was his adoption of Canada as his home in 1936 shortly after his marriage. Professor Coxeter lived enough of his long life in this country that Canadians can claim him as a national hero.

The list of people who came into contact with Professor Coxeter and had an influence on him reads like a "Who's Who" in mathematics. At Cambridge, he was rubbing shoulders with J.E. Littlewood, H.F. Baker and G.H. Hardy. In 1932, he went to Princeton as a Rockefeller Fellow. There he studied under Solomon Lefschetz, Oswald Veblen, John von Neumann, Paul Wigner, and George Pólya. The next year, he returned with another fellowship and was in contact with Albert Einstein and Emmy Noether, and worked closely with Herman Weyl.

I mention all this to give the reader an idea of the richness between the covers of this book. There is a lot more in Part I, and Part II is another treasure trove. The last Appendix is a list of the publications of Professor Coxeter. There are also an amazing 74 pages of Endnotes, detailing the meticulous research which went into this book. Although the author must have been in Professor Coxeter's hair for a considerable period, she is conspicuously absent from her own book.

Having said that, let me indulge in a bit of self-reference of my own. My only meeting with Professor Coxeter was at the International Congress of Mathematics Education in Quebec City in August, 1992. During a lunch cruise on the St. Lawrence River, I sat at the same table as Professor Coxeter and his wife. During the meal, a violinist came and played romantic music. It was the Coxeters' fifty-sixth wedding anniversary! I was playing with a geometric puzzle. Professor Coxeter found it intriguing too, and I was pleased to present him with an impromptu anniversary present.

I also met the author, Siobhan Roberts, once, in connection with another of her literary projects. Having had some first-hand knowledge of her work and a little glimpse of how she works, I have awaited the arrival of my review copy with great excitement, and it has exceeded all my expectations. In the Acknowledgement, she lets slip that she is very good in mathematics too, but that alone cannot explain this wonderful piece of work. I will most certainly buy any of her books, on any subject.

The author remarks that the Hungarian mathematician János Bolyai, at the time of his death, in 1860, had received no recognition for his discovery of non-Euclidean geometry. I am happy to say that we have made sufficient progress to have Professor Coxeter take his rightful place in the history of mathematics in his own lifetime.

On a Theorem of Erdős Concerning Additive Functions

José Luis Ansorena and Juan Luis Varona

Abstract

Erdős proved that every increasing additive function must be a constant multiple of the logarithmic function. We prove a weaker result that assumes that the function is completely additive. In particular, what this paper does show is how wide the gulf is between additive and completely additive functions: proving the result for completely additive functions is very easy, but Erdős's proof for merely additive functions required a formidable effort.

2000 *Mathematics Subject Classification*. Primary 11A25.
Key words: Additive functions, completely additive functions, Erdős.

An *additive* function is defined as a real-valued arithmetic function $f : \mathbb{N} \to \mathbb{R}$ such that $f(nm) = f(n) + f(m)$ for all pairs of coprime integers n and m. If $f(nm) = f(n) + f(m)$ for all $n, m \in \mathbb{N}$, then we say that f is a *completely additive* function.

In [1, Theorem XI, p. 17], Erdős states that if f is an additive function such that $f(n+1) \geq f(n)$ for all $n \in \mathbb{N}$, then $f(n) = C \log n$ for a constant $C \in \mathbb{R}$. Without the hypothesis $f(n+1) \geq f(n)$, this is not true in general, as is shown, for instance, by the completely additive function $\Omega(n)$ defined by $\Omega(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}) = a_1 + a_2 + \cdots + a_k$, where $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the prime decomposition of n. Erdős' Theorem is a deep and interesting result, but its proof is rather complicated; see [3, p. 133] and [2, §8.33 and 8.34, p. 265 and ff.] for adequate comments and additional references, including a proof (due to Moser and Lambek) that simplifies the original proof of Erdős.

In this note, we pose a weaker result, but with a very elementary proof. Also, we show a nice consequence.

Theorem 1. Let f be a completely additive function such that $f(n+1) \geq f(n)$ for all $n \in \mathbb{N}$. Then, there is a real constant $C \geq 0$ such that $f(n) = C \log n$ for all $n \in \mathbb{N}$.

Proof: We claim that, if f and g are two functions satisfying the conditions, they must satisfy

$$f(n)g(2) = f(2)g(n) \quad \forall n \in \mathbb{N} . \tag{1}$$

In particular, this would be true for $g(n) = \log n$. Then, our claim implies that $f(n) = C \log n$ with $C = f(2)/\log 2$ and so the theorem is proved. Thus, we only need check (1).

Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Let us take $l \in \mathbb{N}$ such that $2^{l-1} \leq n^k < 2^l$. Then $(l-1)f(2) \leq kf(n) \leq lf(2)$. The same inequality is true for the function g; we can write it as $-lg(2) \leq -kg(n) \leq -(l-1)g(2)$. Multiplying the first expression by $g(2)$, the second by $f(2)$, adding them, and dividing by k, we get

$$-\frac{1}{k}\,f(2)g(2) \;\leq\; f(n)g(2) - f(2)g(n) \;\leq\; \frac{1}{k}\,f(2)g(2)\,.$$

Since this happens for every $k \in \mathbb{N}$, equation (1) follows and the proof is complete. ∎

As a consequence, let us establish the following result:

Theorem 2. Let $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$ be an increasing and completely additive function. Then, f is the zero function.

Proof: Let us suppose that f is not the zero function. By Theorem 1, there is a real constant $C > 0$ such that $f(n) = C \log n$ for all $n \in \mathbb{N}$. If f takes only integer values, then $f(n)/f(m) = (\log n)/(\log m)$ for all integers n, $m \geq 2$. Therefore, $(\log n)/(\log m) = a/b$ with a, $b \in \mathbb{N}$. This implies that $n^b = m^a$. But this is impossible if n and m have no common prime factor. ∎

Remark. By using Erdős's Theorem, we see that the same proof serves to establish the corresponding result for additive functions.

For completeness, let us show that, without using Theorem 1, another proof of Theorem 2 can be given. Let us begin with

Lemma. Let f be an increasing and completely additive function. Then, $f(1) = 0$. Moreover, if f is not the zero function, it satisfies $f(n) > 0$ for all $n > 1$, and f is strictly increasing.

Proof: We have $f(n) = f(1 \cdot n) = f(1) + f(n)$, and thus $f(1) = 0$. If f is not the zero function, there exists $a \in \mathbb{N}$ such that $f(a) \neq 0$. Since $a \geq 1$, it follows that $f(a) \geq f(1) = 0$; hence, $f(a) > 0$. Now, given $n > 1$, there exists k such that $n^k > a$; then $kf(n) = f(n^k) \geq f(a) > 0$, and we have $f(n) > 0$.

Finally, let us suppose that $f(n) = f(m)$ with $n < m$. This is not possible if $n = 1$, because $f(1) = 0$ and $f(m) > 0$; thus, we may assume that $1 < n < m$. Let us take k large enough such that $n^{k+1} < m^k$ (it suffices to take $k > (\log n)/(\log m - \log n)$). Then

$$f(n^k) \;=\; kf(n) \;=\; kf(m) \;=\; f(m^k)\,.$$

Consequently, $f(r) = f(n^k) = f(m^k)$ for every r such that $n^k \leq r \leq m^k$. In particular, $f(n^{k+1}) = f(n^k)$, and so $(k+1)f(n) = kf(n)$, which is impossible, because $f(n) > 0$. ∎

Using this lemma, we get the following.

Second proof of Theorem 2: For every $n \in \mathbb{N}$, there exists an integer k such that $2^{k+1} - 2^k = 2^k > n$. Let us take n intermediate numbers r_i between 2^k and 2^{k+1}; that is,

$$2^k < r_1 < r_2 < \cdots < r_n < 2^{k+1}.$$

Now, let us suppose that f is not the zero function. By the lemma, f is strictly increasing which implies that

$$\begin{aligned} kf(2) &= f(2^k) < f(r_1) < f(r_2) < \cdots < f(r_n) < f(2^{k+1}) \\ &= (k+1)f(2) = kf(2) + f(2). \end{aligned}$$

Then, by the pigeonhole principle, we have $f(2) > n$. But this happens for every $n \in \mathbb{N}$, which is absurd. ∎

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José Luis Ansorena
Departamento de Matemáticas
y Computación
Universidad de La Rioja
Calle Luis de Ulloa s/n
26004 Logroño, Spain
joseluis.ansorena@dmc.unirioja.es

Juan Luis Varona
Departamento de Matemáticas
y Computación
Universidad de La Rioja
Calle Luis de Ulloa s/n
26004 Logroño, Spain
jvarona@dmc.unirioja.es
http://www.unirioja.es/dptos/dmc/jvarona/welcome.html

PROBLEMS

Solutions to problems in this issue should arrive no later than **1 March 2008**. *An asterisk* $(\star)$ *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

We inadvertently credited Panos E. Tsaoussoglou, Athens, Greece as the proposer of 3225 [2007 : 112, 115]. The actual proposer was George Tsapakidis, Agrinio, Greece. We apologize to both parties for this error.

3221. Correction. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC be a triangle with sides $a \geq b \geq c$ opposite the angles A, B, C, respectively. Let AH be perpendicular to the side BC with H on BC. Set $m = BH$ and $n = CH$. Prove that $a(bm + cn) - bc(b + c)$ is positive, negative, or zero according as $\angle A$ is obtuse, acute, or right-angled.

3251. *Proposed by Michel Bataille, Rouen, France.*

Let u_1, u_2, and u_3 be any real numbers. Prove that

$$\frac{1}{6}\sum_{i=1}^{3}\left[\cos^2(u_i - u_{i+1}) + \cos^2(u_i + u_{i+1})\right] \geq (\cos u_1 \cos u_2 \cos u_3)^2 + (\sin u_1 \sin u_2 \sin u_3)^2 \,,$$

where the subscripts in the summation are taken modulo 3.

3252. *Proposed by Michel Bataille, Rouen, France.*

Let $\mathcal{S}$ be a set of complex 2×2 matrices such that, for all A, B, $C \in \mathcal{S}$, we have $ABCAB = C$.

(a) Show that $(ABC)^n = A^nB^nC^n$ for all positive integers n and all matrices A, B, $C \in \mathcal{S}$.

(b) Give an example of such a set $\mathcal{S}$ containing at least three matrices with two of them non-commuting.

3253. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\log_e(e^\pi - 1)\log_e(e^\pi + 1) + \log_\pi(\pi^e - 1)\log_\pi(\pi^e + 1) \;<\; e^2 + \pi^2 \,.$$

3254. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $\mathcal{C}$ be a convex figure in the plane. A *diametrical chord* AB of $\mathcal{C}$ parallel to the direction vector $\overrightarrow{v}$ is a chord of $\mathcal{C}$ of maximal length parallel to the direction vector $\overrightarrow{v}$.

Prove that if every diametrical chord of $\mathcal{C}$ bisects the area enclosed by $\mathcal{C}$, then $\mathcal{C}$ must be centro-symmetric.

3255. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Prove that, as the points A, B, C move over the surface of an ellipsoid centred at O while the lines OA, OB, OC stay mutually perpendicular, the plane ABC remains tangent to a fixed sphere.

3256. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

A *bicentric* quadrilateral (also called a *chord-tangent* quadrilateral) is a quadrilateral that is simultaneously inscribed in one circle and circumscribed about another.

Let $ABCD$ be a bicentric quadrilateral in which there are no parallel sides. Suppose that the circumscribed and inscribed circles of $ABCD$ have centres O and I, respectively. Let AC meet BD at E. Join the points of tangency on the opposite sides of the quadrilateral, thus obtaining two lines which intersect at a point T.

Prove that O, E, T, and I are collinear. When do the points E and T coincide? (Compare 2978 [2004 : 429, 432; 2005 : 470–472].)

3257. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Find the number of ordered pairs (A, B) of subsets of $\{1, 2, \dots, 13\}$ such that $|A \cup B|$ is even.

3258★. *Proposed by Alper Cay, Uzman Private School, Kayseri, Turkey.*

Let ABC be a triangle with $\angle ABC = 80^\circ$. Let BD be the angle bisector of $\angle ABC$ with D on AC. If $AD = DB + BC$, determine $\angle A$, using a purely geometric argument.

3259. *Proposed by Neven Jurič, Zagreb, Croatia.*

Is it possible to find a cubic polynomial P such that, for any positive integer n, the polynomial $\underbrace{P \circ P \circ \cdots \circ P}_{n \text{ times}}$ has exactly 3^n distinct real roots? Find one, if possible, or show that none exists.

3260. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let a, b be distinct positive real numbers such that $(a-1)(b-1) \geq 0$. Prove that

$$a^b + b^a \;\geq\; 1 + ab + (1-a)(1-b) \cdot \min\{1, ab\}\,.$$

3261. *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

The Fibonacci numbers F_n and Lucas numbers L_n are defined by the following recurrences:

$$\begin{aligned} F_0 &= 0\,, \quad F_1 = 1\,, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1}\,, \quad \text{for } n \geq 1; \\ L_0 &= 2\,, \quad L_1 = 1\,, \quad \text{and} \quad L_{n+1} = L_n + L_{n-1}\,, \quad \text{for } n \geq 1. \end{aligned}$$

Prove that

$$\sum_{n=1}^{\infty} \frac{\arctan\left(\frac{1}{L_{2n}}\right)\arctan\left(\frac{1}{L_{2n+2}}\right)}{\arctan\left(\frac{1}{F_{2n+1}}\right)} \leq \frac{4}{\pi}\arctan(\beta)\left(\arctan(\beta)+\frac{1}{3}\right),$$

where $\beta = \frac{1}{2}(\sqrt{5}-1)$.

3262. *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

Let m be an integer, $m \geq 2$, and let a_1, a_2, $\dots$, a_m be positive real numbers. Evaluate the limit

$$L_m = \lim_{n\to\infty} \frac{1}{n^m} \int_1^e \prod_{k=1}^{m} \ln\bigl(1 + a_k x^n\bigr)\,dx\,.$$

. . . .

3221. Correction. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit ABC un triangle de côtés $a \geq b \geq c$ opposés respectivement aux angles A, B et C. Soit AH la perpendiculaire au côté BC avec H sur BC. Posons $m = BH$ et $n = CH$. Montrer que $a(bm+cn) - bc(b+c)$ est positif, négatif ou nul, suivant que l'angle A est obtus, aigu ou droit.

3251. *Proposé par Michel Bataille, Rouen, France.*

Soit u_1, u_2 et u_3 trois nombres réels arbitraires. Montrer que

$$\frac{1}{6}\sum_{i=1}^{3}\bigl[\cos^2(u_i - u_{i+1}) + \cos^2(u_i + u_{i+1})\bigr] \geq (\cos u_1 \cos u_2 \cos u_3)^2 + (\sin u_1 \sin u_2 \sin u_3)^2\,,$$

où, dans la sommation, les indices sont calculés modulo 3.

3252. *Proposé par Michel Bataille, Rouen, France.*

Soit $\mathcal{S}$ un ensemble de matrices 2×2 complexes telles que pour tout A, B et $C \in \mathcal{S}$, on ait $ABCAB = C$.

(a) Montrer que $(ABC)^n = A^nB^nC^n$ pour tous les entiers positifs n et toutes les matrices A, B et $C \in \mathcal{S}$.

(b) Donner un exemple d'un tel ensemble $\mathcal{S}$ contenant au moins trois matrices avec deux d'entre elles ne commutant pas.

3253. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Montrer que

$$\log_e(e^\pi - 1)\log_e(e^\pi + 1) + \log_\pi(\pi^e - 1)\log_\pi(\pi^e + 1) \ < \ e^2 + \pi^2 \,.$$

3254. *Proposé par George Tsintsifas, Thessalonique, Grèce.*

Soit $\mathcal{C}$ une figure plane convexe. Une *corde diamétrale* AB de $\mathcal{C}$ parallèle au vecteur non nul $\overrightarrow{v}$ est une corde de $\mathcal{C}$ parallèle au vecteur $\overrightarrow{v}$ et de longueur maximale.

Montrer que si toute corde diamétrale de $\mathcal{C}$ sépare la portion bornée par $\mathcal{C}$ en deux parties d'aire égale, alors $\mathcal{C}$ doit posséder un centre de symétrie.

3255. *Proposé par George Tsintsifas, Thessalonique, Grèce.*

Montrer que si les points A, B et C se déplacent au-dessus de la surface d'une ellipsoïde centrée en O et que les droites OA, OB et OC restent mutuellement perpendiculaires, alors le plan ABC reste tangent à une sphère fixe.

3256. *Proposé par Václav Konečný, Big Rapids, MI, USA.*

Un quadrilatère *bicentrique* est un quadrilatère qui possède à la fois un cercle incrit et un cercle circonscrit .

Soit $ABCD$ un quadrilatère bicentrique sans côtés parallèles. Soit respectivement I et O les centres des cercles incrit et circonscrit. Soit E le point d'intersection de AC et BD. Si on relie par des droites les points de tangence des côtés opposés du quadrilatère, elles se coupent en un point T.

Montrer que O, E, T et I sont colinéaires. Quand les points E et T coincident-ils ? (Comparer avec 2978 [2004 : 429, 432 ; 2005 : 470–472].)

3257. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

Trouver le nombre de paires ordonnées (A, B) de sous-ensembles de $\{1, 2, \ldots, 13\}$ telles que $|A \cup B|$ soit pair.

3258★. *Proposé par Alper Cay, Uzman Private School, Kayseri, Turkey.*

Soit ABC un triangle dont l'angle ABC vaut 80°. Soit BD la bissectrice de l'angle ABC, avec D sur AC. Si $AD = DB + BC$, trouver l'angle A en utilisant un argument purement géométrique.

3259. *Proposé par Neven Jurič, Zagreb, Croatie.*

Est-il possible de trouver un polynôme cubique P tel que, pour tout entier positif n, le polynôme $\underbrace{P \circ P \circ \cdots \circ P}_{n \text{ fois}}$ possède exactement 3^n racines réelles distinctes ? En trouver une, si possible, ou montrer qu'il n'en existe aucune.

3260. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit a et b deux nombres réels positifs distincts et tels que $(a-1)(b-1) \geq 0$. Montrer que

$$a^b + b^a \;\geq\; 1 + ab + (1-a)(1-b) \cdot \min\{1, ab\}\,.$$

3261. *Proposé par Ovidiu Furdui, étudiant, Western Michigan University, Kalamazoo, MI, É-U.*

Les nombres de Fibonacci F_n et les nombres de Lucas L_n sont définis par les récurrences :

$$\begin{array}{lllll} F_0 = 0\,, & F_1 = 1\,, & \text{et} & F_{n+1} = F_n + F_{n-1}\,, & \text{pour } n \geq 1\,; \\ L_0 = 2\,, & L_1 = 1\,, & \text{et} & L_{n+1} = L_n + L_{n-1}\,, & \text{pour } n \geq 1. \end{array}$$

Montrer que

$$\sum_{n=1}^{\infty} \frac{\arctan\left(\dfrac{1}{L_{2n}}\right)\arctan\left(\dfrac{1}{L_{2n+2}}\right)}{\arctan\left(\dfrac{1}{F_{2n+1}}\right)} \;\leq\; \frac{4}{\pi}\arctan(\beta)\left(\arctan(\beta) + \frac{1}{3}\right),$$

où $\beta = \frac{1}{2}(\sqrt{5}-1)$.

3262. *Proposé par Ovidiu Furdui, étudiant, Western Michigan University, Kalamazoo, MI, É-U.*

Soit m un entier, $m \geq 2$, et soit $a_1, a_2, \ldots, a_m$ des nombres réels positifs. Calculer la limite

$$L_m \;=\; \lim_{n\to\infty} \frac{1}{n^m} \int_1^e \prod_{k=1}^{m} \ln\bigl(1 + a_k x^n\bigr)\, dx\,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3102. [2006 : 44, 47] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

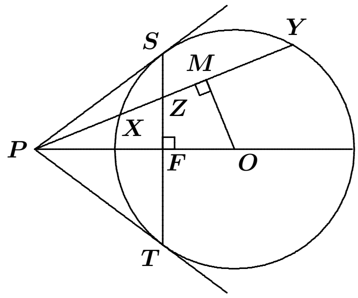
Let D be the mid-point of the side BC of $\triangle ABC$. Let E and F be the projections of B onto AC and C onto AB, respectively. Let P be the point of intersection of AD and EF. Show that, if $AD = \frac{\sqrt{3}}{2} BC$, then P is the mid-point of AD.

III. *Solution by Nobutaka Shigeki, Kitakyusyu City, Fukuoka, Japan, modified by the editor.*

First we will establish a lemma.

Lemma. Let PS and PT be the tangents to a circle at S and T from an exterior point P, as shown is the figure. Let X and Y lie on the circle and be collinear with P. If Z is the point of intersection of ST and XY, then

$$\frac{1}{PX} + \frac{1}{PY} = \frac{2}{PZ}. \qquad (1)$$



Proof: Let M be the mid-point of XY, let O be the centre of the circle, and let F be the intersection of PO with ST. Then

$$\frac{1}{PX} + \frac{1}{PY} = \frac{PX + PY}{PX \cdot PY} = \frac{2PM}{PX \cdot PY}.$$

Therefore, equation (1) is equivalent to $\frac{2PM}{PX \cdot PY} = \frac{2}{PZ}$; that is

$$PM \cdot PZ = PX \cdot PY. \qquad (2)$$

Clearly, the points Z, F, O, and M are concyclic. Thus,

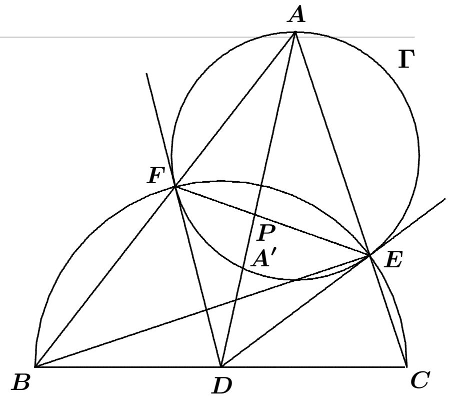
$$PM \cdot PZ = PF \cdot PO = PS^2,$$

because $\triangle PSF$ is similar to $\triangle POS$. Since $PS^2 = PX \cdot PY$, we have (2) and thus (1). ∎

Now we turn our attention to the given problem. Clearly, E and F are the intersections of the circle having BC as diameter with the lines AC and AB, respectively. It follows that $\angle ABC = 180^\circ - \angle FEC = \angle AEF$. On the other hand, since $\triangle BDF$ is isosceles, we have $\angle ABC = \angle BFD$. Thus, $\angle AEF = \angle BFD$. Similarly, $\angle AFE = \angle DEC$.

Let Γ be the circumcircle of $\triangle AEF$. By the Tangent-Chord Theorem, we see that DF is tangent to Γ at F and that DE is tangent to Γ at E. Let A' be the second point of intersection of DA with Γ. Applying the above lemma to Γ, we obtain

$$\frac{2}{DP} = \frac{1}{DA'} + \frac{1}{DA} . \qquad (3)$$



Let $r = DB = DE = DF$. Then $DA' \cdot DA = r^2$. Since we are given that $AD = \sqrt{3}r$, we deduce that $DA' = r/\sqrt{3}$. From (3), we have

$$\frac{2}{DP} = \frac{\sqrt{3}}{r} + \frac{1}{\sqrt{3}\,r} .$$

Therefore, $DP = (\sqrt{3}/2)r = \frac{1}{2}DA$, which means that P is the mid-point of AD.

[*Ed.:* By refining the argument at the end of the proof, one can show that $AD = \frac{\sqrt{3}}{2}BC$ if and only if P is the mid-point of AD.

The above proof assumes that $\triangle ABC$ is acute-angled. However, if there is an obtuse angle at B or at C, the result is still valid. The above proof extends to this case by simply modifying the argument used to show that DE and DF are tangent to Γ.]

3137. [2006 : 173, 176] *Proposed by Tina Balfour and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Find all solutions in non-negative integers to the following Diophantine equations:

(a) $5^m + 3^m = 2^k$; (b) ★ $5^m + 3^n = 2^k$.

(a) *Composite of similar solutions by Brian D. Beasley, Presbyterian College, Clinton, SC, USA; and David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.*

Note first that there are no solutions when $k = 0$. It is also clear that $(m,k) = (0,1)$ and $(1,3)$ are solutions. We now show that there are no other solutions.

Suppose $m \geq 2$. Then $k \geq 6$. Since $2^k \equiv 0 \pmod{16}$ for $k \geq 4$, we have $5^m + 3^m \equiv 0 \pmod{16}$.

However, the least non-negative residues of 5^m modulo 16 for $m \geq 1$ are 5, 9, 13, and 1, which repeat in cycles of length 4, while those of 3^m are 3, 9, 11, and 1, which also repeat in cycles of length 4. Consequently, $5^m + 3^m \equiv 8 \pmod{16}$ or $5^m + 3^m \equiv 2 \pmod{16}$, and our claim follows.

(b) *Solution by Mercedes Sánchez Benito, Universidad Complutense, Madrid, Spain, Óscar Ciaurri Ramírez, Universidad de La Rioja, Logroño, Spain, and Manuel Benito Muñoz and Emilio Fernández Moral, IES Sagasta, Logroño, Spain, modified by the editor.*

If n is even, we have $5^m + 3^n \equiv 1^m + (-1)^n \equiv 2 \pmod 4$. Since $2^k \equiv 2 \pmod 4$ if and only if $k = 1$, the unique solution for n even is $m = n = 0$ and $k = 1$.

Let n be odd. For $m = 0$, we have to find solutions to $1 + 3^n = 2^k$. However, Leo Hebreus (or Levi ben Gerson, 14^{th} century) proved that for all $n > 2$, the integer $3^n \pm 1$ has an odd divisor; hence, the unique solution of $1 + 3^n = 2^k$ for $m = 0$ and n odd is $n = 1$ and $k = 2$.

Now we assume that $m > 0$. By considering the equation modulo 3, we obtain $(-1)^m \equiv (-1)^k \pmod 3$, which implies that m and k have the same parity. On the other hand, by examining the equation modulo 5, we get

$$2^k \equiv (-2)^n \equiv -2^n \equiv \pm 2 \pmod 5 ,$$

since n is odd. This implies that k is odd (and then so is m).

Now suppose that $m \geq 3$ and $n \geq 3$ (which means that $k \geq 7$). Setting $A = 22276800 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$, we checked by a computer program that there are no solutions of $5^m + 3^n = 2^k$ modulo A for odd exponents $m \geq 3$, $n \geq 3$, and $k > 7$ (the checking is a "finite" problem, since $5^{51} \equiv 5^3 \pmod A$, $3^{243} \equiv 3^3 \pmod A$, and $2^{127} \equiv 2^7 \pmod A$). Therefore, we must have either $m = 1$ or $n = 1$.

Let $n = 1$. We must look for solutions of $5^m + 3 = 2^k$ (this problem was proposed on the XXII Spanish Mathematical Olympiad). Using the modulus $B = 65792 = 2^8 \cdot 257$, we again used a computer to search for solutions modulo B (again the checking is a "finite" problem, since $5^{256} \equiv 1 \pmod B$ and $2^{25} \equiv 2^9 \pmod B$; furthermore, 9 is the smallest power of 2 where the remainders modulo B begin to repeat). The computer program yielded the following four cases for $n = 1$ and $m > 0$:

$$(m, k) \in \{(1, 3), (3, 7)\} .$$

Since the values for k lie in the non-periodic set of remainders of powers of 2 modulo B, we see that $k = 1$ or $k = 7$. This gives us the solutions $(m, n, k) = (1, 1, 3)$ and $(m, n, k) = (3, 1, 7)$. Furthermore, any other solutions must have $m \equiv 1 \pmod B$ or $m \equiv 3 \pmod B$. Since the smallest values for m other than 1 or 3 are significantly too large to have a solution, these are the only solutions for $n = 1$.

Lastly, we will examine $m = 1$ and $n \geq 3$. This time, we use the modulus $C = 2^6 \cdot 3^4 \cdot 17$ for our computer check. Once more this becomes a finite problem since $3^{21} \equiv 3^5 \pmod C$ and $2^{223} \equiv 2^7 \pmod C$; furthermore, 5 and 7 are the smallest powers of 3 and 2, respectively, where the remainders begin to repeat. The only solution modulo C that the program generated was $(n, k) = (3, 5)$. This yields the solution $(m, n, k) = (1, 3, 5)$. Since the powers on both 2 and 3 are in the non-periodic set of remainders of their respective powers, there are no further solutions.

In conclusion, there are exactly five solutions to $5^m + 3^n = 2^k$, namely:

$$(m,n,k) \in \{(0,0,1),\ (0,1,2),\ (1,1,3),\ (3,1,7),\ (1,3,5)\}\,.$$

Part (a) *also solved by MICHEL BATAILLE, Rouen, France; ROY BARBARA, Lebanese University, Fanar, Lebanon; MERCEDES SÁNCHEZ BENITO, Universidad Complutense, Madrid, Spain, ÓSCAR CIAURRI RAMÍREZ, Universidad de La Rioja, Logroño, Spain, and MANUEL BENITO MUNOZ ̃and EMILIO FERNÁNDEZ MORAL, IES Sagasta, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.*

Beasley conjectured that the equation in part (b) *has exactly the five solutions which are determined above.*

The reason for the late featuring of this solution is that we wanted to have the computer solution properly analyzed. We apologize for this delay. We would appreciate if our readers could find a proof for the result which is independent of computer verification.

3139. [2006 : 238, 240; 2007 : 242] *Proposed by Michel Bataille, Rouen, France.*

Let ε be the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$. Two parallel tangents to ε intersect a third tangent at $M_1(x_1,y_1)$ and $M_2(x_2,y_2)$. Show that the value of

$$\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}-1\right)$$

is independent of the chosen tangents.

II. *Solution by J.A. Thas, Ghent University, Ghent, Belgium.*

The desired result is a consequence of properties of projective coordinates interpreted in the affine plane. Our conic defines a *scalar product* between the points $M_1=(x_1,y_1)$ and $M_2=(x_2,y_2)$ by

$$\langle M_1,M_2\rangle \;=\; \frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1\,.$$

There is likewise a scalar product defined by the dual conic (composed of the tangents to the conic) between pairs of lines: if $L_i=[u_i,v_i,w_i]$ represent the lines $u_ix+v_iy+w_i=0$ for $i=1$ and $i=2$, then

$$[L_1,L_2] \;=\; a^2u_1u_2+b^2v_1v_2-w_1w_2\,.$$

A pair of points or a pair of lines are conjugate if and only if their scalar product is zero. One easily shows that the line joining M_1 to M_2 is tangent to the conic if and only if

$$\langle M_1,M_1\rangle\langle M_2,M_2\rangle-\langle M_1,M_2\rangle^2 \;=\; 0\,. \tag{1}$$

(See, for example, H.S.M. Coxeter, *The Real Projective Plane*, 3rd edition, formula 12.76 on page 188, for the details.)

In this notation, we are required to show that $\langle M_1, M_1\rangle\langle M_2, M_2\rangle$ is independent of the chosen tangents. We will show that, for all choices of the three tangents, $\langle M_1, M_1\rangle\langle M_2, M_2\rangle = 1$. In view of (1) above, we have only to show that $\langle M_1, M_2\rangle^2 = 1$.

Consider the parallelogram formed by the given parallel tangents together with the third tangent and the tangent parallel to it. Because the three diagonals of any quadrilateral circumscribed about a conic form a self-polar triangle (this is the dual of Theorem 6.43 on page 78 of the Coxeter book cited above), the diagonals of our parallelogram, namely $y_1x - x_1y = 0$ and $y_2x - x_2y = 0$, are conjugate. This tells us that

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} = 0,$$

as desired.

3151. [2006 : 304, 306] *Proposed by Mª Jesús Villar Rubio, Santander, Spain.*

(a) Let $r_1 < 0 < r_2 < r_3$ be the real roots of $8x^3 - 6x + \sqrt{3} = 0$. Prove that

$$r_3^2 = 4r_2^2 - 4r_2^4 \quad \text{and} \quad r_1^2 = 4r_3^2 - 4r_3^4 .$$

(b) Let $s_1 < 0 < s_2 < s_3$ be the real roots of $8x^3 - 6x + 1 = 0$. Prove that

$$r_1^2 + s_2^2 = 1, \quad s_1^2 + r_2^2 = 1, \quad \text{and} \quad r_3^2 + s_3^2 = 1 .$$

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC, USA.

(a) Set $x = \sin\theta$. The given equation $8x^3 - 6x + \sqrt{3} = 0$ may be rewritten as

$$\frac{\sqrt{3}}{2} = -4x^3 + 3x = -4\sin^3\theta + 3\sin\theta = \sin(3\theta).$$

Then $3\theta = \frac{\pi}{3} + 2\pi k$ or $3\theta = \frac{2\pi}{3} + 2\pi k$, for any integer k. Restricting θ to the interval $[0, 2\pi]$, we find that

$$\theta \in \left\{\frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}, \frac{13\pi}{9}, \frac{14\pi}{9}\right\}.$$

Thus,

$$r_1 = \sin\frac{13\pi}{9} = -\sin\frac{4\pi}{9}, \quad r_2 = \sin\frac{\pi}{9}, \quad \text{and} \quad r_3 = \sin\frac{2\pi}{9}.$$

Using the identity $\sin^2 2\theta = 4\sin^2\theta - 4\sin^4\theta$, we have

$$r_3^2 = \sin^2\frac{2\pi}{9} = 4\sin^2\frac{\pi}{9} - 4\sin^4\frac{\pi}{9} = 4r_2^2 - 4r_2^4$$

and

$$r_1^2 = \sin^2\frac{4\pi}{9} = 4\sin^2\frac{2\pi}{9} - 4\sin^4\frac{2\pi}{9} = 4r_3^2 - 4r_3^4 .$$

(b) Set $x = \cos\theta$. The given equation $8x^3 - 6x + 1 = 0$ may be rewritten as

$$-\frac{1}{2} = 4x^3 - 3x = 4\cos^3\theta - 3\cos\theta = \cos(3\theta)\,.$$

Then $3\theta = \frac{2\pi}{3} + 2\pi k$ or $3\theta = \frac{4\pi}{3} + 2\pi k$, for any integer k. Restricting θ to the interval $[0, 2\pi]$, we find that

$$\theta \in \left\{\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}\right\}.$$

Thus,

$$s_1 = \cos\frac{8\pi}{9} = -\cos\frac{\pi}{9}\,, \quad s_2 = \cos\frac{4\pi}{9}\,, \quad \text{and} \quad s_3 = \cos\frac{2\pi}{9}\,.$$

The desired result follows from the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ applied to the above values of r_i and s_i.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; QUANG CAO MINH, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ÓLÍNA SIGURGEIRSDÓTTIR, student, Auburn University, Montgomery, AL, USA; BIN ZHAO, student, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; and the proposer. Most solutions were based on the identity for $\cos 3\theta$.

3152. [2006 : 304, 307] *Proposed by Michel Bataille, Rouen, France.*

Let $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ be real numbers such that $\sum_{i=1}^{n} x_i = 0$ and $\sum_{i=1}^{n} x_i^2 = 1$. Find the minimum and maximum of $\sum_{i=1}^{n} |x_i|$.

Essentially the same solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Kee-Wai Lau, Hong Kong, China.

We have

$$\left(\sum_{i=1}^{n} |x_i|\right)^2 = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |x_i||x_j| \geq 1 + \left|\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} x_i x_j\right|$$

$$= 1 + \left|\left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2\right| = 2\,.$$

Equality holds for $x_1 = 1/\sqrt{2} = -x_2$ and $x_3 = x_4 = \cdots = x_n = 0$. It follows that the minimum is $\sqrt{2}$.

For the maximum, suppose that m of the x_i's are non-negative and $n-m$ are non-positive, for $1 \leq m \leq n$. Without loss of generality, we assume that $x_1, \ldots, x_m$ are non-negative and $x_{m+1}, \ldots, x_n$ are non-positive. By Schwarz's Inequality,

$$\sum_{i=1}^{m} x_i^2 \;\geq\; \frac{1}{m}\left(\sum_{i=1}^{m} x_i\right)^2$$

and

$$\sum_{i=m+1}^{n} x_i^2 \;\geq\; \frac{1}{n-m}\left(\sum_{i=m+1}^{n} x_i\right)^2 \;=\; \frac{1}{n-m}\left(\sum_{i=1}^{m} x_i\right)^2 .$$

It follows that

$$1 \;= \sum_{i=1}^{n} x_i^2 \;\geq\; \left(\frac{1}{m}+\frac{1}{n-m}\right)\left(\sum_{i=1}^{m} x_i\right)^2 \;=\; \frac{n}{m(n-m)}\left(\sum_{i=1}^{m} x_i\right)^2 .$$

Hence, $\sum\limits_{i=1}^{n} |x_i| = 2\sum\limits_{i=1}^{m} x_i \leq \frac{2}{\sqrt{n}}\sqrt{m(n-m)}$.

For any real number r, the quadratic function $x(r-x)$ has a maximum $r^2/4$ at $x = r/2$. Thus, for $n = 2k$, where k is a positive integer, we have $m(n-m) \leq k^2$ and

$$\sum_{i=1}^{n} |x_i| \;\leq\; \frac{2k}{\sqrt{n}} \;=\; \sqrt{n}\,.$$

Equality holds for $x_1 = x_2 = \cdots = x_k = -x_{k+1} = \cdots = -x_n = 1/\sqrt{n}$.

For $n = 2k+1$, where k is a positive integer,

$$m(n-m) \;\leq\; \max\{k(2k+1-k),(k+1)(2k+1-k-1)\} \;=\; k(k+1)\,.$$

Hence,

$$\sum_{i=1}^{n} |x_i| \;\leq\; \frac{2\sqrt{k(k+1)}}{\sqrt{n}} \;=\; \sqrt{n-\frac{1}{n}}\,.$$

Equality holds for

$$x_1 \;=\; x_2 \;=\; \cdots \;=\; x_k \;=\; \sqrt{\frac{k+1}{k(2k+1)}}$$

$$\text{and} \qquad x_{k+1} \;=\; \cdots \;=\; x_n \;=\; -\sqrt{\frac{k}{(k+1)(2k+1)}}\,.$$

We conclude that the maximum is $\sqrt{n+\dfrac{(-1)^n-1}{2n}}$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect submission.

3153. [2006 : 304, 307] *Proposed by Michel Bataille, Rouen, France.*

For which integers n does the equation

$$\frac{3xy-1}{x+y} = n$$

have a solution in integers x and y?

Essentially the same solution by Roy Barbara, Lebanese University, Fanar, Lebanon; Brian D. Beasley, Presbyterian College, Clinton, SC, USA; Kee-Wai Lau, Hong Kong, China; Joel Schlosberg, Bayside, NY, USA; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let k be any integer. If $n = 3k$, then the given equation becomes $3xy - 3k(x+y) = 1$, which has no solutions. If $n = 3k+1$, then $x = k$ and $y = -(3k^2+k+1)$ is a solution. If $n = 3k-1$, then $x = k$ and $y = 3k^2 - k + 1$ is a solution. Hence, the equation has a solution if and only if n is not a multiple of 3.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

3154 [2006 : 304, 307] *Proposed by Challa K.S.N.M. Sankar, Andhra-pradesh, India.*

(a) If $\beta > 1$ is a real constant, determine the number of possible real solutions of the equation

$$x - \beta\log_2 x = \beta - \beta\ln\beta\,.$$

(b) If $\alpha_1 < \alpha_2$ are two positive real solutions of the equation in (a), and if x_1 and x_2 are any two real numbers satisfying $\alpha_1 \le x_1 < x_2 \le \alpha_2$, prove that, for all λ such that $0 < \lambda < 1$,

$$\lambda\log_2 x_1 + (1-\lambda)\log_2 x_2 \ \ge\ \ln\bigl(\lambda x_1 + (1-\lambda)x_2\bigr)\,.$$

Determine when equality occurs.

Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina, modified and expanded by the editor.

(a) Consider the function $f(x) = x - \beta\log_2 x - \beta + \beta\ln\beta$ for $x > 0$. We have $f'(x) = 1 - \beta/(x\ln 2)$; whence, $x = \beta/(x\ln 2)$ is the only critical value. Since $f'(x) < 0$ for $0 < x < \beta/\ln 2$ and $f'(x) > 0$ for $x > \beta/\ln 2$, we see that f is decreasing on $(0, \beta/\ln 2)$ and increasing on $(\beta/\ln 2, \infty)$.

Thus, f has a relative minimum at $x = \beta/\ln 2$.

Note that $\lim_{x\to 0^+} f(x) = \infty$ and that

$$\lim_{x\to\infty} f(x) \;=\; -\beta + \beta\ln\beta + \lim_{x\to\infty} x\left(1 - \frac{\beta\log_2 x}{x}\right) = \infty\,,$$

since $\lim_{x\to\infty} \dfrac{\beta\log_2 x}{x} = 0$.

We now show that $f(\beta/\ln 2) < 0$.

Since $f(\beta/\ln 2) = \beta/\ln 2 - \beta\log_2(\beta/\ln 2) - \beta + \beta\ln\beta$, it suffices to show that $1/\ln 2 - \log_2(\beta/\ln 2) - 1 + \ln\beta < 0$, which is equivalent in succussion to

$$\begin{aligned} 1 - (\ln 2)\big(\log_2(\beta/\ln 2)\big) - \ln 2 + (\ln 2)(\ln\beta) &< 0\,,\\ 1 - \ln 2 - \ln(\beta/\ln 2) + (\ln 2)(\ln\beta) &< 0\,,\\ 1 - \ln 2 - \big(\ln\beta - \ln(\ln 2)\big) + (\ln 2)(\ln\beta) &< 0\,,\\ 1 - \ln 2 + \ln(\ln 2) - (1 - \ln 2)(\ln\beta) &< 0\,, \end{aligned}$$

which is true since $(1-\ln 2)(\ln\beta) > 0$ and $1-\ln 2+\ln(\ln 2) < 0$. Therefore, f has *exactly* two real roots, α_1 and α_2, such that $0 < \alpha_1 < \alpha_2$. That is, the given equation has exactly two real solutions.

(b) From part (a) we see that $f(x) \le 0$ for all $x \in [\alpha_1, \alpha_2]$, where $0 < \alpha_1 < \beta/\ln 2 < \alpha_2$. Thus, $\log_2 x \ge (x/\beta) - 1 + \ln\beta$.

Since $0 < \lambda < 1$, it follows that

$$\begin{aligned} &\lambda\log_2 x_1 + (1-\lambda)\log_2 x_2 \\ &\qquad\ge\; \lambda\left(\frac{x_1}{\beta} - 1 + \ln\beta\right) + (1-\lambda)\left(\frac{x_2}{\beta} - 1 + \ln\beta\right) \\ &\qquad=\; \frac{1}{\beta}\big(\lambda x_1 + (1-\lambda)x_2\big) - 1 + \ln\beta\,. \end{aligned} \tag{1}$$

Next we show that, for all $t > 0$,

$$t - \beta + \beta\ln\beta \;\ge\; \beta\ln t\,. \tag{2}$$

Let $g(t) = t - \beta + \beta\ln\beta - \beta\ln t$. Then $g'(t) = 1 - \beta/t$ showing that $t = \beta$ is the only critical value. Since $g''(t) = \beta/t^2 > 0$, we see that $g(\beta) = 0$ is a relative as well as the absolute minimum of g. Hence, $g(t) \ge 0$ for all $t > 0$ and (2) follows.

In particular, for $t = \lambda x_1 + (1-\lambda)x_2$, we obtain

$$\lambda x_1 + (1-\lambda)x_2 - \beta + \ln\beta \;\ge\; \beta\ln\big(\lambda x_1 + (1-\lambda)x_2\big)\,. \tag{3}$$

From (1) and (3), we then have

$$\lambda\log_2 x_1 + (1-\lambda)\log_2 x_2 \;\ge\; \ln\big(\lambda x_1 + (1-\lambda)x_2\big)\,.$$

Equality occurs if and only if $x_1 = \alpha_1$, $x_2 = \alpha_2$, and $\lambda\alpha_1 + (1-\lambda)\alpha_2 = \beta$, which yields $\lambda = \dfrac{\alpha_2 - \beta}{\alpha_2 - \alpha_1}$. Since $f(\beta) = \beta(\ln\beta - \log_2\beta) < 0$, we see that $\alpha_1 < \beta < \dfrac{\beta}{\ln 2} < \alpha_2$, which is consistent with the assumption $0 < \lambda < 1$.

Also solved by the proposer.

3155. [2006 : 304, 307] *Proposed by Virgil Nicula, Bucharest, Romania.*

In $\triangle ABC$, let D, E, F be the intersections of the altitudes from A, B, C to the sides BC, CA, AB, respectively. Let H be the orthocentre of $\triangle ABC$, let L be the intersection of AT and the line through B perpendicular to BC, and let T be the intersection of BE and DF.

Show that $BL = BC$ if and only if $\angle ACB = 45^\circ$.

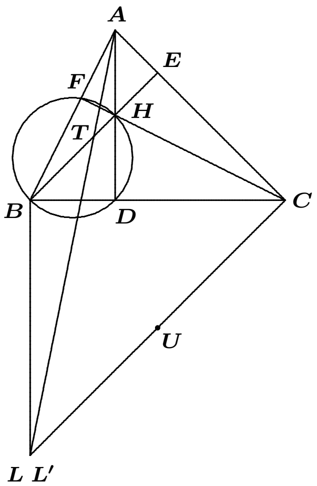
Solution by Michel Bataille, Rouen, France.

We modify the requirement of the problem to be:

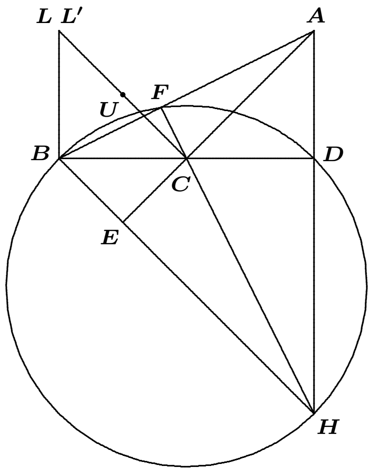
Show that $BL = BC$ if and only if $\angle ACB = 45^\circ$ or $\angle ACB = 135^\circ$, and that AT is the polar of C with respect to γ.

Note that B, F, H, and D lie on the circle γ with diameter BH and that AT is the polar with respect to γ. We call L' the point of intersection of the perpendiculars to BC at B and to AC at C.

First, suppose that $\angle ACB = 45^\circ$ or $\angle ACB = 135^\circ$. Then using the facts that $BE \parallel CL'$ and $\angle BCL' = 45^\circ$, we see that $\triangle CBL'$ and $\triangle BDH$ are isosceles right triangles, with right angles at B and D, respectively. Let U be the mid-point of CL'. Then, $\angle UBH = 90^\circ$, so that UB is tangent to γ at B and the circle $\gamma' = (BCL')$ is orthogonal to γ. Since CL' is a diameter of γ', the points C and L' are conjugate with respect to γ. Hence, L' is on the polar AT of C, and thus, $L = L'$ and $BL = BC$.



$\angle ACB = 45^\circ$



$\angle ABC = 135^\circ$

Also solved by APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and VÁCLAV KONEČNÝ, Big Rapids, MI, USA.

The following solvers only considered the case $\angle ACB = 45^\circ$: ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3156. [2006 : 305, 307] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let Γ be the circumcircle of $\triangle ABC$. Let M be an interior point on the side AB, and let N be an interior point on the side AC. Let D be an intersection point of MN with Γ. Prove that

$$\left|\frac{MB}{MA}\cdot\frac{AC}{DB}-\frac{NC}{NA}\cdot\frac{AB}{DC}\right| \;=\; \frac{BC}{DA}\,.$$

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let $P = AD\cap BC$ and $Q = MD\cap BC$. [*Editor's comment:* Woo deals explicitly with the case where D is opposite B on arc AC of the circumcircle, and B lies between Q and C. With the use of directed distances and directed angles, we can avoid special cases except when P or Q is at infinity; these possibilities are easily handled using continuity arguments.] By Menelaus' Theorem applied to the transversal QMD of $\triangle BAP$ and to the transversal QND of $\triangle CAP$,

$$\frac{BM}{MA} \;=\; -\frac{BQ}{QP}\cdot\frac{PD}{DA} \qquad\text{and}\qquad \frac{CN}{NA} \;=\; -\frac{CQ}{QP}\cdot\frac{PD}{DA}\,.$$

Because $\triangle PCA \sim \triangle PDB$, we have

$$\frac{AC}{BD} \;=\; \frac{PC}{PD}\,;$$

similarly, $\triangle PBA \sim \triangle PDC$ implies that

$$\frac{AB}{CD} \;=\; \frac{PB}{PD}\,.$$

It follows that

$$\begin{aligned}
\frac{BM}{MA}\cdot\frac{AC}{BD}-\frac{CN}{NA}\cdot\frac{AB}{CD} &= -\frac{BQ}{QP}\cdot\frac{PD}{DA}\cdot\frac{AC}{BD}+\frac{CQ}{QP}\cdot\frac{PD}{DA}\cdot\frac{AB}{CD}\\
&= -\frac{BQ}{QP}\cdot\frac{PD}{DA}\cdot\frac{PC}{PD}+\frac{CQ}{QP}\cdot\frac{PD}{DA}\cdot\frac{PB}{PD}\\
&= \frac{-BQ\cdot PC+CQ\cdot PB}{QP\cdot DA}\\
&= \frac{-BQ\cdot PC+(CB+BQ)\cdot(PC+CB)}{QP\cdot DA}\\
&= \frac{CB\cdot(PC+CB+BQ)}{QP\cdot DA}\\
&= \frac{CB\cdot PQ}{QP\cdot DA} \;=\; \frac{BC}{DA}\,,
\end{aligned}$$

as desired.

Also solved by MICHEL BATAILLE, Rouen, France; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

Nicula provided a selection of familiar and not so familiar special cases of his result:

- *If* $BN \cap CM = G$ *(the centroid), then* $\left|\frac{b}{DB} - \frac{c}{DC}\right| = \frac{a}{DA}$.
- *If* $BN \cap CM = I$ *(the incentre), then* $\left|\frac{1}{DB} - \frac{1}{DC}\right| = \frac{1}{DA}$.
- *If* $BN \cap CM = H$ *(the orthocentre), then* $\left|\frac{\cos B}{DB} - \frac{\cos C}{DC}\right| = \frac{|\cos A|}{DA}$.

3157. [2006 : 305, 308] *Proposed by Mihály Bencze, Brasov, Romania.*

Let p be a fixed odd prime number. Let $\alpha(n)$ denote the largest integer k such that p^k is an integral divisor of $1^1 \cdot 2^2 \cdot 3^3 \cdots n^n$. Prove that

$$\lim_{n\to\infty} \frac{\alpha(n)}{n^2} \;=\; \frac{1}{2(p-1)}\,.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

In order to get an idea of the behavior of $\alpha(n)$, we proceed very much like Legendre in his reasoning for the exponent of p in the prime decomposition of $n!$. The result then is

$$\alpha(n) \;=\; p\left(1+2+\cdots+\left\lfloor\frac{n}{p}\right\rfloor\right) + p^2\left(1+2+\cdots+\left\lfloor\frac{n}{p^2}\right\rfloor\right) + p^3\left(1+2+\cdots+\left\lfloor\frac{n}{p^3}\right\rfloor\right) + \cdots\,.$$

In a more concise form,

$$\alpha(n) \;=\; \sum_{j=1}^{N(n)} \frac{p^j}{2}\left\lfloor\frac{n}{p^j}\right\rfloor\left(\left\lfloor\frac{n}{p^j}\right\rfloor+1\right)\,,$$

where $N(n) = \lfloor \ln n/\ln p\rfloor$. We have

$$\left(\frac{n}{p^j}-1\right)\left(\frac{n}{p^j}\right) \;<\; \left\lfloor\frac{n}{p^j}\right\rfloor\left(\left\lfloor\frac{n}{p^j}\right\rfloor+1\right) \;\le\; \left(\frac{n}{p^j}\right)\left(\frac{n}{p^j}+1\right)\,.$$

Multiplying by p^j yields

$$n\left(\frac{n}{p^j}-1\right) \;<\; p^j\left\lfloor\frac{n}{p^j}\right\rfloor\left(\left\lfloor\frac{n}{p^j}\right\rfloor+1\right) \;\le\; n\left(\frac{n}{p^j}+1\right)\,.$$

Thus,

$$\frac{1}{2}\left(\left(\sum_{j=1}^{N(n)}\frac{1}{p^j}\right)-\frac{N(n)}{n}\right) < \frac{\alpha(n)}{n^2} \le \frac{1}{2}\left(\left(\sum_{j=1}^{N(n)}\frac{1}{p^j}\right)+\frac{N(n)}{n}\right).$$

Letting $n \to \infty$ and noting that $\lim\limits_{n\to\infty}\frac{N(n)}{n} = 0$, we get

$$\lim_{n\to\infty}\frac{\alpha(n)}{n^2} = \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{p^j} = \frac{1}{2(p-1)}.$$

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3158. [2006 : 305, 308] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $E = \{(x,y) \in \mathbb{N}\times\mathbb{N} \mid x+y \text{ is a perfect square}\}$, and let $N(n)$ be the size of the set $\{(x,y) \in E \mid x \le n \text{ and } y \le n\}$, for $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty}\frac{N(n)}{n\sqrt{n}} = \frac{4}{3}(\sqrt{2}-1).$$

Solution by Joel Schlosberg, Bayside, NY, USA, modified by the editor.

For each $n \in \mathbb{N}$ and $k \in \mathbb{N}$, let $\varphi(n,k)$ be the number of integer pairs (x,y) with $1 \le x$, $y \le n$ and $x+y=k$. Then $N(n) = \sum\limits_{i=1}^{\infty}\varphi(n,i^2)$.

To evaluate $\varphi(n,k)$, we first observe that if $1 \le x$, $y \le n$, then $2 \le x+y \le 2n$. It follows that $\varphi(n,k) = 0$ unless $2 \le k \le 2n$. If $2 \le k \le n+1$, then the equation $x+y=k$ is satisfied by the pairs $(1,k-1)$, $(2,k-2)$, ..., $(k-1,1)$; hence, $\varphi(n,k) = k-1$. If $n+1 \le k \le 2n$, then the equation $x+y=k$ is satisfied by $(k-n,n)$, $(k-n+1,n-1)$, ..., $(n,k-n)$; hence, $\varphi(n,k) = 2n+1-k$. Thus,

$$\varphi(n,k) = \begin{cases} k-1 & \text{if } 2 \le k \le n+1, \\ 2n+1-k & \text{if } n+1 \le k \le 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for each $i \in \mathbb{N}$,

$$\varphi(n,i^2) = \begin{cases} i^2-1 & \text{if } 2 \le i \le I_1, \\ 2n+1-i^2 & \text{if } I_1 \le k \le I_2, \\ 0 & \text{otherwise,} \end{cases}$$

where $I_1 = \lfloor\sqrt{n+1}\rfloor$ and $I_2 = \left\lfloor\sqrt{2n}\right\rfloor$.

Now

$$\begin{aligned} N(n) &= \sum_{i=1}^{\infty} \varphi(n,i^2) = \sum_{i=2}^{I_1}(i^2-1) + \sum_{i=I_1+1}^{I_2}(2n+1-i^2) \\ &= \sum_{i=1}^{I_1} i^2 - I_1 + (2n+1)(I_2-I_1) - \sum_{i=I_1+1}^{I_2} i^2 \\ &= 2\sum_{i=1}^{I_1} i^2 - \sum_{i=1}^{I_2} i^2 + 2n(I_2-I_1) + I_2 - 2I_1 \\ &= \frac{I_1(I_1+1)(2I_1+1)}{3} - \frac{I_2(I_2+1)(2I_2+1)}{6} \\ &\qquad + 2n(I_2-I_1) + I_2 - 2I_1 \\ &= \frac{\sqrt{n}\cdot\sqrt{n}\cdot 2\sqrt{n}}{3} - \frac{\sqrt{2n}\cdot\sqrt{2n}\cdot 2\sqrt{2n}}{6} + 2n(\sqrt{2n}-\sqrt{n}) + O(n) \\ &= n\sqrt{n}\left(\frac{2}{3} - \frac{2\sqrt{2}}{3} + 2\sqrt{2} - 2\right) + O(n) \\ &= \frac{4}{3}n\sqrt{n}\left(\sqrt{2}-1\right) + O(n)\,. \end{aligned}$$

Thus,

$$\frac{N(n)}{n\sqrt{n}} = \frac{4}{3}\left(\sqrt{2}-1\right) + O\left(\frac{1}{\sqrt{n}}\right),$$

and therefore,

$$\lim_{n\to\infty} \frac{N(n)}{n\sqrt{n}} = \frac{4}{3}\left(\sqrt{2}-1\right),$$

as claimed.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3159. [2006 : 305, 308] *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be a positive integer, and let γ be Euler's constant. Prove that

$$\gamma - \frac{1}{48n^3} < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) < \gamma - \frac{1}{48(n+1)^3}\,.$$

Solution by the proposer.

For each positive integer n, let

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) + \frac{1}{48n^3}\,.$$

Then $x_{n+1} - x_n = f(n)$, where

$$f(x) = \frac{1}{x+1} - \ln\left(x + \frac{3}{2} + \frac{1}{24(x+1)}\right) + \ln\left(x + \frac{1}{2} + \frac{1}{24x}\right) + \frac{1}{48(x+1)^3} - \frac{1}{48x^3}\,.$$

We have $f'(x) > 0$ for $x > 0$. [*Ed:* Using a computer algebra system, we get $f'(x) = \frac{2656x^6+10096x^5+15008x^4+10836x^3+3870x^2+652x+37}{16x^4(x+1)^4(24x^2+12x+1)(24x^2+60x+37)}$.] Furthermore, $\lim\limits_{x\to\infty} f(x) = 0$. Therefore, $f(x) < 0$ for all $x > 0$, which implies that the sequence $\{x_n\}_{n=1}^{\infty}$ is strictly decreasing. Since $\lim\limits_{n\to\infty} x_n = \gamma$, we must have $x_n > \gamma$ for all n. This proves the left inequality.

For each positive integer n, let

$$y_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) + \frac{1}{48(n+1)^3}\,.$$

Then $y_{n+1} - y_n = g(n)$, where

$$g(x) = \frac{1}{x+1} - \ln\left(x + \frac{3}{2} + \frac{1}{24(x+1)}\right) + \ln\left(x + \frac{1}{2} + \frac{1}{24x}\right) + \frac{1}{48(x+2)^3} - \frac{1}{48(x+1)^3}\,.$$

We have $g'(x) < 0$ for $x > 0$. [*Ed:* Using a computer algebra system, we get $g'(x) = -\frac{8864x^7+72336x^6+247520x^5+456204x^4+483110x^3+288492x^2+86997x+9472}{16x(x+1)^4(x+2)^4(24x^2+12x+1)(24x^2+60x+37)}$.] Furthermore, $\lim\limits_{x\to\infty} g(x) = 0$. Therefore, $g(x) > 0$ for all $x > 0$, which shows that the sequence $\{y_n\}_{n=1}^{\infty}$ is strictly increasing. Since $\lim\limits_{n\to\infty} y_n = \gamma$, we must have $y_n < \gamma$ for all n. This proves the right inequality.

Also solved by PAUL BRACKEN and N. NADEAU, University of Texas, Edinburg, TX, USA; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. There was one incomplete solution.

3160. [2006 : 305, 308] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let $\triangle ABC$ have altitude AD and orthocentre H. Let E be the mid-point of AD and M the mid-point of BC.

(a) If $AD = BC$, prove that $HM = HE$.

(b) Is the converse of (a) true?

I. *A composite of similar solutions by Roy Barbara, Lebanese University, Fanar, Lebanon; and Geoffrey A. Kandall, Hamden, CT, USA.*

Dealing with parts (a) and (b) together, we prove that $HM = HE$ if and only if $AD = BC$. We introduce coordinates with $D(0,0)$, $E(0,1)$, and $A(0,2)$ on the y-axis, while $B(b,0)$ and $C(c,0)$ define the x-axis for real numbers $c > b$. It follows that M is the point $\left(\frac{1}{2}(b+c),0\right)$ and the line CH, passing through C and perpendicular to AB, has the equation $y = \frac{1}{2}bx - \frac{1}{2}bc$. The y-intercept of CH is $H(0,-\frac{1}{2}bc)$. We thus have

$$\begin{aligned} HM^2 - HE^2 &= \left(\frac{b+c}{2}\right)^2 + \frac{b^2c^2}{4} - \left(1+\frac{bc}{2}\right)^2 \\ &= \frac{b^2+c^2}{4} - \frac{bc}{2} - 1 \\ &= \frac{1}{4}\big((b-c)^2 - 4\big) = \frac{1}{4}(BC^2 - AD^2)\,. \end{aligned}$$

Thus, $HM = HE$ if and only if $BC = AD$.

II. *A composite of similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Titu Zvonaru, Comănești, Romania.*

Because $\triangle ABD \sim \triangle CHD$ (since $\angle BAD = 90° - \angle ABD = \angle HCD$), we deduce that $\dfrac{AD}{CD} = \dfrac{BD}{HD}$. In terms of signed segments, this equality becomes

$$DA \cdot HD = DB \cdot DC\,.$$

We therefore have

$$\begin{aligned} HM^2 - HE^2 &= HD^2 + DM^2 - HE^2 \\ &= HD^2 + (DB+BM)^2 - (HD+DE)^2 \\ &= HD^2 + \left(DB + \tfrac{1}{2}BC\right)^2 - \left(HD + \tfrac{1}{2}DA\right)^2 \\ &= \tfrac{1}{4}(BC^2 - AD^2) - DA\cdot HD + DB(DB+BC) \\ &= \tfrac{1}{4}(BC^2 - AD^2) - DA\cdot HD + DB\cdot DC \\ &= \tfrac{1}{4}(BC^2 - AD^2)\,. \end{aligned}$$

Thus, $HM = HE$ if and only if $AD = BC$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (a second solution); MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA (2 solutions); JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

One of Konečný's solutions considered the family of all triangles ABC with fixed base BC for which $AD = BC$. The locus of orthocentres H as A moves along the line parallel to BC at the fixed distance of $AD = BC$ is a parabola whose focus is the mid-point M of BC, latus rectum is BC, and directrix is the locus of the mid-point E of AD. The equality of HM and HE is then equivalent to a basic property of conics: From any point on a parabola, the distance to the directrix equals the distance to the focus.

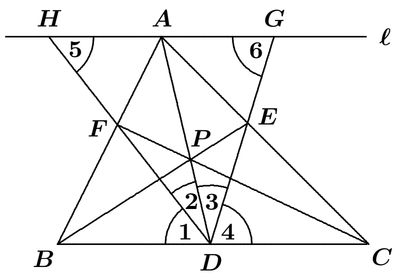
3161. [2006 : 305, 308] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let D be a point on the side BC of $\triangle ABC$, and let P be an arbitrary point on the segment AD. Let BP meet AC at E and CP meet AB at F.

(a) If $AD \perp BC$, prove that $\angle BDF = \angle CDE$.

(b) Is the converse of (a) true?

Solution by Geoffrey A. Kandall, Hamden, CT, USA, modified by the editor.



Let ℓ be the line through vertex A parallel to the side BC. Let DE and DF meet ℓ at points G and H, respectively. Let 1, 2, 3, 4, 5, and 6 denote the angles BDF, FDA, EDA, CDE, AHD, and AGD, respectively. Clearly, $\angle 5 = \angle 1$ and $\angle 6 = \angle 4$.

First we show that $HA = AG$. Triangles AHF and BFD are similar, which implies that $\frac{AF}{FB} = \frac{HA}{BD}$. Likewise, triangles AGE and CDE are similar, yielding $\frac{CE}{EA} = \frac{DC}{AG}$. Applying Ceva's Theorem to $\triangle ABC$, we get

$$1 = \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{HA}{BD} \cdot \frac{BD}{DC} \cdot \frac{DC}{AG} = \frac{HA}{AG}.$$

Thus, $HA = AG$.

(a) If $AD \perp BC$, then triangles DAH and DAG are congruent, implying that $\angle 1 = \angle 4$.

(b) If $\angle 1 = \angle 4$, then $\angle 5 = \angle 6$, which implies that $\triangle DGH$ is isosceles (with $DG = DH$). The median DA in $\triangle DGH$ is therefore also the altitude from D to GH. This makes $AD \perp \ell$ and thus also $AD \perp BC$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Konečný and Malikić pointed out that part (a) *of the problem is known; it had appeared, in almost identical form, as the fifth problem of the Canadian Mathematical Olympiad in 1994 with solution published in Crux* [1994 : 189]. *Konečný also supplied an earlier reference* (A Survey of Geometry, *Howard Eves, 1972, Allyn and Bacon, p.86*).

3162. [2006 : 306, 308] *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Determine all integer solutions (x, y) of the equation

$$x^5 + y^7 = 2004^{1007} .$$

Solution by Kee-Wai Lau, Hong Kong, China.

We show that the equation has no integer solutions by considering residues modulo 71.

Since 71 is a prime and $(16, 71) = 1$, we have $(16^{14})^{70} \equiv 1 \pmod{71}$ by Fermat's Little Theorem. Hence,

$$\begin{aligned} 2004^{1007} &\equiv 16^{1007} = (16^{14})^{70}(16^{27}) \equiv 16^{27} = 4096^9 \\ &\equiv 49^9 \equiv 117649^3 \equiv 2^3 = 8 \ (\pmod{71}) . \end{aligned}$$

With the help of a computer, we find that the quintic residues (mod 71) are

$$0,\ 1,\ 20,\ 23,\ 26,\ 30,\ 32,\ 34,\ 37,\ 39,\ 41,\ 45,\ 48,\ 51,\ \text{and}\ 70$$

and the septic residues (mod 71) are

$$0,\ 1,\ 5,\ 14,\ 17,\ 25,\ 46,\ 54,\ 57,\ 66,\ \text{and}\ 70.$$

It follows by tedious but straightforward calculations that the residues of $x^5 + y^7 \pmod{71}$ are precisely those k where $0 \le k \le 70$ such that $k \notin \{8, 10, 11, 60, 61, 63\}$.

Since the residue 8 is missing, we conclude that the given equation has no integer solutions.

Also solved (using essentially the same argument) by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; and the proposer. There was also an incomplete solution.

The proposer remarked that 71 is the smallest prime for which this proof works.

3163. [2006 : 394, 396] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Calculate

$$\lim_{n\to\infty} \ln\left(\prod_{k=1}^{n}\left(\frac{k^2+n^2}{n^2}\right)^k\right)^{\frac{1}{n^2}} .$$

Composite of virtually identical solutions by those solvers identified by an asterisk beside their names in the list at the end.

Let $S_n = \ln\left(\prod_{k=1}^{n}\left(\frac{k^2+n^2}{n^2}\right)^k\right)^{\frac{1}{n^2}}$. Then

$$S_n = \frac{1}{n^2}\sum_{k=1}^{n} k\ln\left(\frac{k^2+n^2}{n^2}\right) = \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\ln\left(1+\frac{k^2}{n^2}\right) ,$$

a Riemann sum associated with the continuous function $f(x) = x\ln(1+x^2)$ and the regular partition $\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$ of the interval $[0,1]$.

Hence,

$$\begin{aligned}\lim_{n\to\infty} S_n &= \int_0^1 x\ln(1+x^2)\,dx = \frac{1}{2}\int_1^2 \ln u\,du = \frac{1}{2}(u\ln u - u)\Big|_1^2 \\ &= \frac{1}{2}\big((2\ln 2 - 2) - (-1)\big) = \frac{1}{2}(-1+2\ln 2)\,.\end{aligned}$$

*Solved by *MICHEL BATAILLE, Rouen, France; *DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; *MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; *CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; *PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; *JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; *ALEXANDROS SYGELAKIS, student, University of Crete, Heraklion, Greece; *PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. Two incorrect solutions were also received.*